第五屆培正數學邀請賽
The 5th Pui Ching Invitational Mathematics Competition

決賽（高中組）
Final Event (Senior Secondary)

時限：2 小時
Time allowed: 2 hours

參賽者須知：
Instructions to Contestants:

1. 本卷共設甲、乙兩部分，總分爲 100 分。
   This paper is divided into Section A and Section B. The total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

甲部（75分）

Section A (75 marks)

1. 考慮算式「$2 \div 2 \div 2 \div 2 \div 2$」。這算式的值是 $\frac{1}{8}$。若在算式上加上括號（次數不限），則算式的最大可能值是多少？（3分）

   Consider the expression '$2 \div 2 \div 2 \div 2 \div 2$'. The value of this expression is $\frac{1}{8}$. If parentheses are added onto the expression (we can add as many as we wish), what is the greatest possible value of the expression? (3 marks)

2. 一個 n 邊形的每隻內角的大小（以「度」爲單位時）全是 7 的倍數。求 n 的最小可能值。（3分）

   The size (in degrees) of every interior angle of an n-sided polygon is divisible by 7. Find the smallest possible value of n. (3 marks)

3. 設 k 爲正整數。若圓 $x^2 + y^2 + 2x + 4y + (7 - 2k) = 0$ 穿過直角坐標平面上全部四個象限，求 k 的最小可能值。（3分）

   Let k be a positive integer. If the circle $x^2 + y^2 + 2x + 4y + (7 - 2k) = 0$ passes through all four quadrants on the rectangular coordinate plane, find the smallest possible value of k. (3 marks)

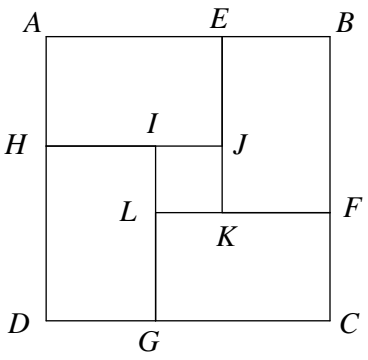
4. 小權、小仁、小年和小龍四人玩擲硬幣比賽。每兩人之間都比賽一場。已知每場比賽雙方勝出的機會均等，而且沒有和局，求他們四人勝出場數皆不同的概率。（4分）

   Donald, Rafael, Henry and Stephen have a coin-tossing competition. One match is played between every two of them. Given that the two players in a match are equally likely to win and there are no ties, find the probability that the numbers of matches won by the four players are all different. (4 marks)

5. 已知 B 和 C 是實數。小華做練習題時，發現 $x^4 + (B - 2C)x^2 + 2B + 8 = 0$ 和 $y^4 + (3B + C)y^2 + 4B + 2C = 0$ 這兩個方程都剛好有三個不同的實數解。求 C 的值。（4分）

   Given B and C are real numbers. When working on exercises, Roy found that both equations $x^4 + (B - 2C)x^2 + 2B + 8 = 0$ and $y^4 + (3B + C)y^2 + 4B + 2C = 0$ have exactly three different real solutions. Find the value of C. (4 marks)

6. 圖中，四個全等而且面積爲 1 的長方形 *AEJH*、*BFKE*、*CGLF* 和 *DHIG*，與及正方形 *IJKL* 拼成一個大正方形 *ABCD*。若 *C*、*L*、*H* 共線，求 *ABCD* 的面積。（4 分）



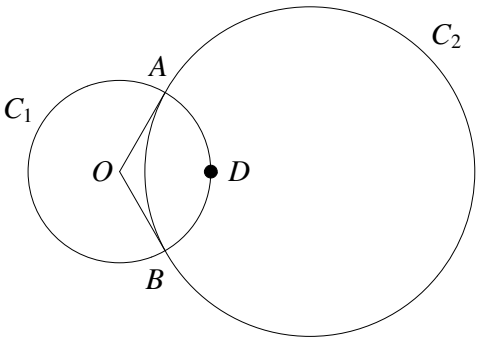
In the figure, four identical rectangles *AEJH*, *BFKE*, *CGLF* and *DHIG*, each with area 1, together with the square *IJKL*, are combined to form a large square *ABCD*. If *C*, *L*, *H* are collinear, find the area of *ABCD*. (4 marks)

7. 求 $\int_0^{\frac{\pi}{2}} \frac{1+\sin x+\sin 2x}{\sin x+\cos x}\,\mathrm{d}x$ 的值。（5 分）

Evaluate $\int_0^{\frac{\pi}{2}} \frac{1+\sin x+\sin 2x}{\sin x+\cos x}\,\mathrm{d}x$. (5 marks)

8. 圓 C_1 的圓心爲 *O*，半徑爲 1。另一圓 C_2 與 C_1 相交於 *A* 和 *B*，而 *OA* 和 *OB* 皆與 C_2 相切。*D* 是 C_1 的圓周上的一點，且位於 C_2 的內部。若弧 *ADB* 的長度是 C_1 的周長的三分之一，求兩圓重疊部分的面積。（5 分）

The circle C_1 is centred at *O* and has radius 1. Another circle C_2 meets C_1 at *A* and *B* such that both *OA* and *OB* are tangent to C_2. *D* is a point on the circumference of C_1 and in the interior of C_2. If the length of the arc *ADB* is equal to one-third of the perimeter of C_1, find the area of the region overlapped by the two circles. (5 marks)



9. 當 $x^{2006}+2005$ 除以 $(x+1)^2$ 時，餘式是 $ax+b$。求 $2a-3b$ 的值。（5 分）

If the remainder of $x^{2006}+2005$ when divided by $(x+1)^2$ is $ax+b$, find the value of $2a-3b$. (5 marks)

10. 光明學校及仁愛學校參加了一項比賽，期間兩所學校均須把學生分組。若兩所學校各自把自己的學生分成 24 人一組，則兩所學校各有一些學生剩餘下來，而兩所學校剩餘的學生加起來共有 34 人。同樣地，若兩所學校各自把自己的學生分成 34 人一組，則兩所學校剩餘的學生加起來共有 42 人。現把兩間學校的學生組成聯隊，再分成 51 人一組，會有多少名學生剩餘下來？ （6 分）

Candle Light School and Friendship School take part in a competition, during which they have to divide their students into groups. If each of the two schools divides their own students into groups of 24, then each school has some students left ungrouped, and the sum of the numbers of students left is 34. Similarly, if each of the two schools divides their own students into groups of 34, then the sum of the numbers of students left is 42. If the students from the two schools form a single team and are divided into groups of 51, how many students will be left ungrouped? (6 marks)

11. 已知 r 是正數，而方程 $\cos\theta = \sin r\theta$ 在 $0° \le \theta \le 180°$ 的區間裏剛好有 2006 個解。求 r 的最小可能值。 （6 分）

Given r is a positive number and the equation $\cos\theta = \sin r\theta$ has exactly 2006 solutions in the interval $0° \le \theta \le 180°$. Find the smallest possible value of r. (6 marks)

12. 小明做三角幾何的練習題時，其中一題是「求 $\tan \overline{ABC}°$ 的值」。他不小心地把題目看錯成「求 $\tan \overline{BAC}°$ 的值」，怎料他的答案卻恰巧與正確答案相同。已知 $A > B > 0$，$\overline{ABC}$ 這個數有多少個可能值？（題中 $\overline{XYZ}$ 表示百位是 X，十位是 Y，個位是 Z 的三位數。） （6 分）

When working on exercises in trigonometry, Mike misread the question 'Find the value of $\tan \overline{ABC}°$' as 'Find the value of $\tan \overline{BAC}°$'. However, the answer turned out to be the same as the correct answer. Given that $A > B > 0$, how many different possible values are there for the number $\overline{ABC}$? (In the question $\overline{XYZ}$ denotes the three-digit number with hundreds digit X, tens digit Y and unit digit Z.) (6 marks)

13. 某城市的電話號碼全爲八位數字，而所有電話的鍵盤款式均如圖所示。該城市並規定所有電話號碼的相鄰數字，必須爲電話鍵盤內的相鄰數字（例如：85256321）。問有多少個可用的電話號碼？ （7 分）

7	8	9
4	5	6
1	2	3

In a city, all telephone numbers have 8 digits and all telephones have a keyboard in the form as shown. Furthermore, consecutive digits in a telephone number must be adjacent digits on the keyboard (e.g. 85256321). How many different possible telephone numbers are there? (7 marks)

14. 培正中學於 1889 年創校。已知 $1889 = 33^2 + 2(20)^2$，且存在正整數 x、y，使得 $1889^2 = x^2 + 2y^2$。求 x。（7 分）

Pui Ching Middle School was founded in 1889. Given that $1889 = 33^2 + 2(20)^2$ and there exist positive integers x, y such that $1889^2 = x^2 + 2y^2$, find x. (7 marks)

15. 如圖所示，六位學生 A、B、C、D、E 和 F 依次序圍成圓圈玩遊戲，裁判把六張分別寫上 0 至 5 的遊戲卡分派給六人。每個人只可以看到自己和他旁邊兩人的卡上的數字。每個人所得的分數就是他所看到的三張卡上的數字之積。以下是他們於較後時間的對話。

裁判問所有學生：「你們知道誰的分數最高嗎？」

A、B、C 和 D 四人同時回答：「我知道了。」

然後 E 和 F 二人亦說：「聽過你們四人的說話後，我也知道了。」

假設學生都是聰明的（即當他們有足夠的資料便一定可以作出推論）。已知 B 的卡上的數字比 C 大。求 F 的分數。（7 分）

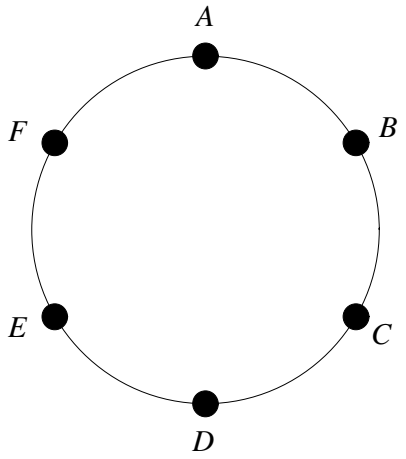
Six students, A, B, C, D, E and F, sit around a circle as shown in the figure. Six cards numbered 0 to 5 are distributed to the six people by a judge. Each person can only see the numbers on his own card and the cards of his two neighbours. The score of each person is the product of the three numbers he can see. Their subsequent conversations are as follows.

'Do you know whose score is the highest?' The judge asked all students.

'Yes, I know.' replied A, B, C and D.

After that, E and F also said, 'Having listened to all four of you, I know now.'

Assume that all students are intelligent (i.e. they can make deductions whenever there is enough information). It is known that the number on B's card is greater than that on C's card. Find the score of F. (7 marks)



乙部（**25** 分）

Section B (25 marks)

細閱以下資料，然後回答第 16 至第 20 題。

Study the following information and answer Questions 16 to 20.

一項世界性足球比賽共有 160 隊參加，比賽共分成三個階段。在第一階段，這 160 隊先分成 8 組，每組 20 隊進行單循環賽，即每隊都會與同組的其他各隊比賽一次。每場勝方得 2 分、負方得 0 分、賽和則各得 1 分。每組得分最高的 4 隊可晉身第二階段，如有同分則以抽簽決定出線隊伍。

第二階段的 32 隊分成 8 組，每組 4 隊進行單循環賽，即每隊都會與同組的其他各隊比賽一次。每場勝方得 3 分、負方得 0 分、賽和則各得 1 分。每組得分最高的 2 隊可晉身第三階段，如有同分則以抽簽決定出線隊伍。

第三階段則採用淘汰賽形式，每場勝方晉級，負方出局。賽和則加時再賽，再和則互射十二碼分勝負。

每隊有 23 名球員，他們被編成 1 至 23 號。每場球賽每隊可派 11 人出賽，並有 3 個換人名額。被換出的球員不能在同一場比賽中再被換入場。

160 teams compete in a worldwide soccer tournament. The tournament consists of three stages. In Stage 1, the 160 teams are divided into 8 groups of 20 teams each. Each team plays exactly one match against every other team in the same group. In a match, the winner gets 2 points and the loser gets 0 point. If a match ends in a tie, both teams get 1 point. The 4 teams with the highest number of points in each group can proceed to Stage 2. In case of a tie, the result will be resolved by drawing lots.

In Stage 2, the 32 teams are divided into 8 groups of 4 teams each. Each team plays exactly one match against every other team in the same group. In a match, the winner gets 3 points and the loser gets 0 point. If a match ends in a tie, both teams get 1 point. The 2 teams with the highest number of points in each group can proceed to Stage 3. In case of a tie, the result will be resolved by drawing lots.

Stage 3 is an elimination stage; the winner in each match can proceed while the loser is eliminated. In case of a tie, an extra time will be played. If the match still ends in a tie, the winner will be determined by penalty kicks.

There are 23 players in each team, numbered 1 to 23. In each match, each team may send 11 players with 3 chances for substitution. Players who have been substituted out cannot be substituted in again in the same match.

16. 每支球隊在整個賽事中最多會進行多少場比賽？（在第三階段中，16 支隊伍會先進行 8 場比賽，8 場的勝方再進行 4 場比賽，如此類推。） （3 分）

What is the maximum number of matches a team may play in the entire tournament? (In Stage 3, the 16 teams will compete in 8 matches, and the 8 winners will then compete in 4 matches, and so on.) (3 marks)

17. 在第一階段的比賽中，一支球隊得到 n 分，並且晉身第二階段。求 n 的最小可能值。 （5 分）

A team obtained n points in Stage 1 and was able to proceed to Stage 2. Find the smallest possible value of n. (5 marks)

18. 在第一階段的某組比賽中，20 隊最後所得的分數互不相同。若該組的比賽共有 k 場賽和，求 k 的最大可能值。 （7 分）

During Stage 1, the 20 teams in a certain group obtained pairwise different scores. If k of the matches in that group resulted in a tie, find the greatest possible value of k. (7 marks)

19. 在第二階段中，32 支球隊按以下程序被分成 8 組：

- 32 隊先被編成 A、B、C、D 四級，每級 8 隊。
- 每次從每級抽出一隊，被抽出的四隊同組。這步驟不斷重覆，直至 8 組全部產生。

已知 R 隊屬 A 級、S 隊屬 B 級、T 隊屬 C 級，那麼 R、S、T 三隊互不同組的概率是多少？ （4 分）

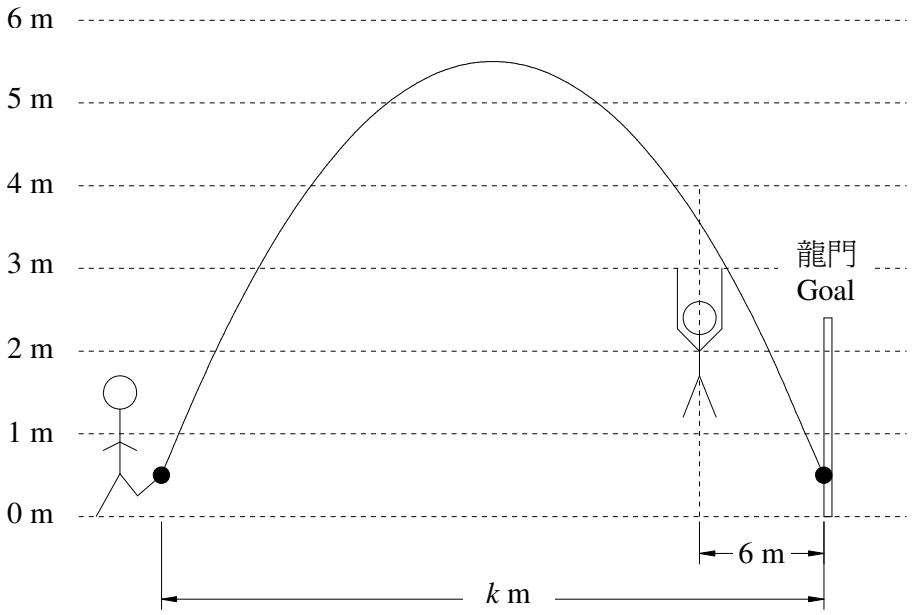
In Stage 2, the 32 teams are put into 8 groups by the following procedure.

- The 32 teams are put into four pots A, B, C and D, with 8 teams in each pot.
- One team is drawn randomly from each pot, and the four drawn teams form a group. This process is repeated until all 8 groups are formed.

Given that Team R is in Pot A, Team S is in Pot B and Team T is in Pot C, find the probability that Teams R, S and T are in pairwise different groups. (4 marks)

20. 在 C 隊對 D 隊的一場比賽中，C 隊的一名前鋒離門 k 米，他看見 D 隊守門員離門達 6 米，於是立即來個「凌空笠射」。他施射時球離地 0.5 米，而球射出後，離地高度最高達 5.4 米，其軌跡呈拋物線（見下圖）。若以鉛垂線爲 y 軸、以球在地面的投影的軌跡爲 x 軸，則該拋物線可在直角坐標平面上以 $y = ax^2 + bx + c$（其中 a、b、c 是常數）表示。守門員雖然看準時間向上跳起，但他雙手離地最高亦只有 3 米，而球下墜時卻在離地 3.8 米處經過。結果這球順利射入，而球入網的一刻離地 0.5 米。求 k。（6 分）

In a match between Team C and Team D, a striker of Team C was k metres from the goal. Seeing that the goalkeeper of Team D was 6 metres from the goal, he made a lob shot in the air immediately. When the shot was made, the ball was 0.5 metre from the ground. After the shot, the maximum height of the ball from the ground was 5.4 metres and the locus of the ball was a parabola (see the figure below). The parabola can be represented as $y = ax^2 + bx + c$ (where a, b, c are constants) on the rectangular coordinate plane with the vertical as y-axis and the locus of the projection of the ball on the ground as x-axis. Although the goalkeeper watched the time well and made a jump vertically from where he stood, his hands could only reach a height of 3 metres while the ball passed at a height of 3.8 metres in its downward trajectory. The shot was a goal and the height of the ball from the ground at the time of entering the goal was 0.5 metre. Find k. (6 marks)



全卷完

END OF PAPER