第十屆培正數學邀請賽

10th Pui Ching Invitational Mathematics Competition

決賽（高中組）

Final Event (Senior Secondary)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

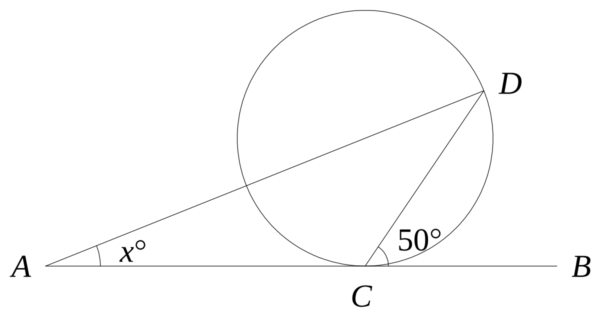
The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 圖中，直線 AB 切圓於 C。D 是圓周上的一點，且 DA 穿過圓心。若 $\angle DCB = 50°$ 而 $\angle DAB = x°$，求 x。

   In the figure, the straight line AB is tangent to the circle at C. D is a point on the circumference such that DA passes through the centre of the circle. If $\angle DCB = 50°$ and $\angle DAB = x°$, find x.



2. 希淳把 20110402 的數字重新排列，得到一個新的八位數，這個八位數可被 5^k 整除。求 k 的最大可能值。

   Helsa rearranged the digits of 20110402 to form a new eight-number number which is divisible by 5^k. Find the greatest possible value of k.

3. 嘉珮寫下了一個 20 項的正整數數列。從第二項起，每項都比之前的一項大 5。各項中最多有幾個是質數？

   Lyanne has written a sequence of 20 positive integers. Starting from the second term, each term is 5 greater than the previous term. What is the maximum number of terms that are prime?

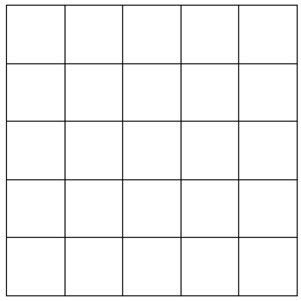
4. 在一個排球比賽中，開始時的計分牌顯示 $0:0$，每球勝出的一方可得 1 分，先得到 25 分的一方勝出（不設「刁時」，即如果打成 $24:24$ 平手，再勝出一球的一方便會勝出）。計分牌顯示的分數有多少個不同的可能？（註：若 $a \neq b$，則 $a:b$ 和 $b:a$ 視爲不同的分數。）

   In a volleyball match, the scoreboard initially shows $0:0$. Each time the winner scores 1 point, and whichever side gets 25 points first wins (no ‘deuce’ is played, i.e. when the score is $24:24$, the side getting the next point wins). How many different scores are possible on the scoreboard? (Note: We regard $a:b$ and $b:a$ to be different scores if $a \neq b$.)

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 在一個遊戲中，參加者需在圖中的 5×5 方格陣的 25 個小格中每次選一個，直至選了五個中心成一直線的小格便算勝出。若一名參加者選了 n 個不同的小格後仍沒有勝出，求 n 的最大可能值。



In a game, the player shall choose one of the 25 small squares at a time in the 5×5 grid shown, and wins when five small squares whose centres lie on the same straight line have been chosen. If a player has not yet won after having chosen n different small squares, find the greatest possible value of n.

6. 設 $f(x) = (2a-6)x^2 + (a^2-9)x + a$ ，其中 a 是實數。若 $\int_0^2 f(x)\,\mathrm{d}x = \int_4^6 f(x)\,\mathrm{d}x = \int_7^9 f(x)\,\mathrm{d}x$ ，求 a 的值。

Let $f(x) = (2a-6)x^2 + (a^2-9)x + a$ where a is a real number. If $\int_0^2 f(x)\,\mathrm{d}x = \int_4^6 f(x)\,\mathrm{d}x = \int_7^9 f(x)\,\mathrm{d}x$, find the value of a.

7. 若實數 y 滿足 $y^y = 10$ ，求 $y\log(\log y)$ 的值。

If the real number y satisfies $y^y = 10$, find the value of $y\log(\log y)$.

8. 若 $1 \le x \le 10$ 和 $1 \le y \le 10$ ，求 $y^2x - 3y(3-x)$ 的最小可能值。

If $1 \le x \le 10$ and $1 \le y \le 10$, find the smallest possible value of $y^2x - 3y(3-x)$.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

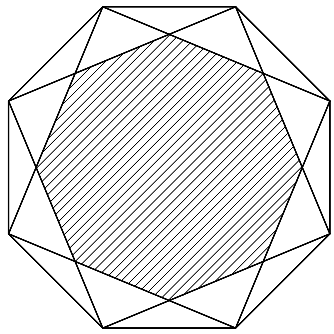
9. 在所示的乘式中，每個字母代表一個由 0 至 9 的不同數字。求乘積 FBF 所代表的三位數。

   In the multiplication shown, each letter represents a different digit from 0 to 9. Find the three-digit number represented by the product FBF.

$$\begin{array}{rccc} & & \mathrm{A} & \mathrm{B} \\ \times & & \mathrm{C} & \mathrm{D} \\ \hline & \mathrm{A} & \mathrm{B} & \\ & \mathrm{A} & \mathrm{E} & \mathrm{F} \\ \hline & \mathrm{F} & \mathrm{B} & \mathrm{F} \end{array}$$

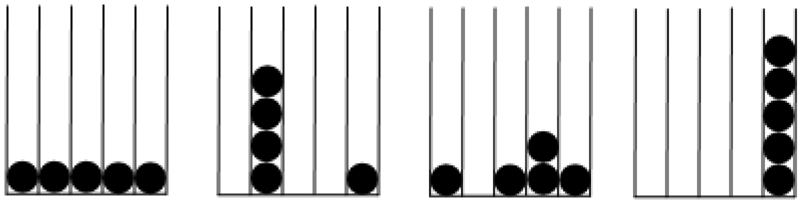
10. 若把一個邊長為 1 的正八邊形的八個頂點梅花間竹地分成兩組，則每組的四個頂點均組成一個正方形。求兩個正方形重疊部分的面積。

    If the eight vertices of a regular octagon of side length 1 are divided into two groups in an alternate manner, the four vertices in each group will form a square. Find the area of the region overlapped by the two squares.



11. 在一個遊戲中，參加者需把 5 個球射進 5 條垂直的坑道。如果 5 個球最終成一直線，參加者便勝出。（下圖顯示了四個例子，當中只有第一個和第四個勝出。）如果射了某數目的球後已經肯定無法勝出，那麼遊戲立即結束，參加者不能再射餘下的球。若每個球都必定會射進其中一條坑道，且射進各坑道的機會均等，求射了 4 球後遊戲便結束的概率。

    In a game, the player has to shoot 5 balls into one of the 5 vertical tracks, and wins if the 5 balls turn out to be collinear. (The figure below shows four examples, among which only the first and fourth outcomes are winning.) If the player is certain not to win after shooting a certain number of balls, the game ends immediately without shooting the remaining balls. If each ball must get into one of the tracks with equal probability, find the probability that the game ends after 4 balls are shot.



12. 2011 個連續正奇數之和是個立方數。若 k 是當中最小的一個奇數，求 k 的最小可能值。

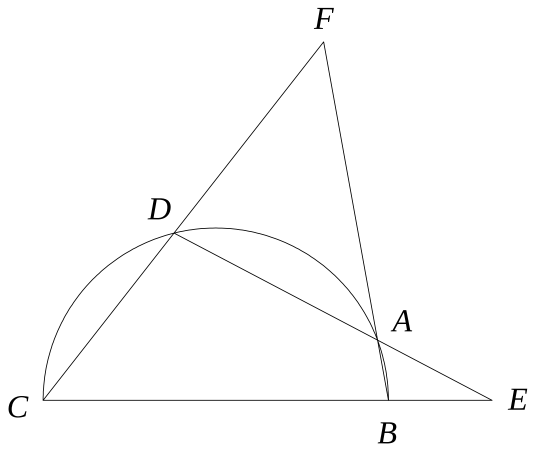
The sum of 2011 consecutive odd positive integers is a cubic number. If k is the smallest of these odd numbers, find the smallest possible value of k.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 圖中，BC 是半圓的直徑，A、D 爲圓周上的兩點。CB 與 DA 延長後交於 E，CD 與 BA 延長後交於 F。若 $BC = 2$、$\angle CED$ = 15° 且 $\angle CFB$ = 45°，求 AE^2，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示（其中 a、b 爲有理數）。

In the figure, BC is the diameter of the semi-circle and A, D are points on the circumference. CB and DA are extended to meet at E, while CD and BA are extended to meet at F. If $BC = 2$, $\angle CED$ = 15° and $\angle CFB$ = 45°, find AE^2 in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where a, b are rational numbers.



14. 浩宏把水注進一個長方體至半滿，該長方體的底是一個面積 28 cm^2 的正方形。當他把長方體沿其中一條底邊傾側 x° 時，水面的面積變成 35 cm^2；當他把長方體再傾側 x° 後，水面的面積變成 n cm^2。求 n。（假設水面一直沒有觸及柱體底部。）

Adrian fills a cuboid to half-full. The base of the cuboid is a square with area 28 cm^2. When he tilts the cuboid for an angle of x° along one of the edges of the base, the area of the water surface is 35 cm^2; when he further tilts the cuboid by another x°, the area of the water surface becomes n cm^2. Find n. (Assume that the water surface does not touch the base of the prism.)

15. 設 f、g 爲二次函數。若方程 $f(g(x)) = 0$ 有 3、1、4 和 p 共四個實根，求 p 的最大可能值。

Let f, g be quadratic functions. If the equation $f(g(x)) = 0$ has four real roots 3, 1, 4 and p, find the greatest possible value of p.

16. 設 $a_1 = 40$ ，並對正整數 n 定義

$$a_{n+1} = \begin{cases} a_n - \frac{1}{n} & \text{若 } a_n{}^2 < 2011 \\ a_n + \frac{1}{n} & \text{若 } a_n{}^2 > 2011 \\ 2011 & \text{若 } a_n{}^2 = 2011 \end{cases}$$

求 $\lim_{n\to\infty} a_n$ 。（換句話說，當 n 很大時， a_n 會趨近甚麼數？）

Let $a_1 = 40$. For positive integer n, we define

$$a_{n+1} = \begin{cases} a_n - \frac{1}{n} & \text{if } a_n{}^2 < 2011 \\ a_n + \frac{1}{n} & \text{if } a_n{}^2 > 2011 \\ 2011 & \text{if } a_n{}^2 = 2011 \end{cases}$$

Find $\lim_{n\to\infty} a_n$. (In other words, to what number will a_n approach when n is large?)

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 求當 $C_2^{1234} + C_6^{1234} + C_{10}^{1234} + \cdots + C_{1230}^{1234} + C_{1234}^{1234}$ 除以 30 時的餘數。

Find the remainder when $C_2^{1234} + C_6^{1234} + C_{10}^{1234} + \cdots + C_{1230}^{1234} + C_{1234}^{1234}$ is divided by 30.

18. 一個袋子中有 1 張紅牌、2 張黃牌和 7 張白牌。小莉每次從袋子中抽出一張牌然後放回袋子，直至抽到紅牌或連續抽到兩張黃牌便停止。求她抽了第五張牌後停止的概率。

In a bag there is 1 red card, 2 yellow cards and 7 white cards. Each time Lily draws one card from the bag and then puts it back into the bag, until she draws a red card or two consecutive yellow cards, at which point she stops. Find the probability that she stops after drawing the fifth card.

19. 在一個遊戲中，參加者不斷投擲一枚勻稱的硬幣，每次擲得正面可得 1 分，擲得反面可得 2 分，累積達 2011 分或以上遊戲便結束。已知一名參加者結束遊戲時得到剛好 2011 分的概率以最簡分數表示爲 $\frac{p}{q}$，求 p 除以 64 時的餘數。

In a game, a participant repeatedly tosses a fair coin, and gets 1 point each time a head is obtained, and 2 points each time a tail is obtained. The game ends when the participant has accumulated 2011 points or above. Given that the probability for a participant to end the game with exactly 2011 points is $\frac{p}{q}$ in lowest term, find the remainder when p is divided by 64.

20. 在一個重組句子遊戲中，參加者需要把 6 張分別寫上「數」、「學」、「很」、「有」、「趣」和「味」的卡片重新排列。遊戲的計分方法如下：正確的排列是「數學很有趣味」，在重新排列卡片後，我們把 6 張排列後的卡片分成最長的連續正確段（即連續地在正確排列中出現的卡片），每個由 k 張卡片組成的最長連續正確段可得 2^k 分。例如：如果排列是「味有趣數學很」，則最長的連續正確段分別是「味」、「有趣」和「數學很」，因此得分是 $2^1+2^2+2^3=14$。若把卡片隨意排列，則最有可能得到的分數是 S，且有 n 個不同的排列可以得到 S 分。求 n。

In a sentence reconstruction game, players had to rearrange 6 cards labelled 'mathematics', 'is', 'a', 'very', 'interesting' and 'subject'. The score is computed as follows. With 'mathematics is a very interesting subject' being the correct order, the 6 rearranged cards are divided into maximal consecutive correct segments (i.e. consecutive cards which appear in the correct order). Each maximal consecutive correct segment consisting of k cards is worth 2^k points. For instance, for the rearrangement 'subject very interesting mathematics is a', the maximal consecutive correct segments are 'subject', 'very interesting' and 'mathematics is a', and hence the score is $2^1+2^2+2^3=14$. By a random rearrangement of the cards, the most probable score is S with n different rearrangements leading to such score. Find n.

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