第十屆培正數學邀請賽

10th Pui Ching Invitational Mathematics Competition

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

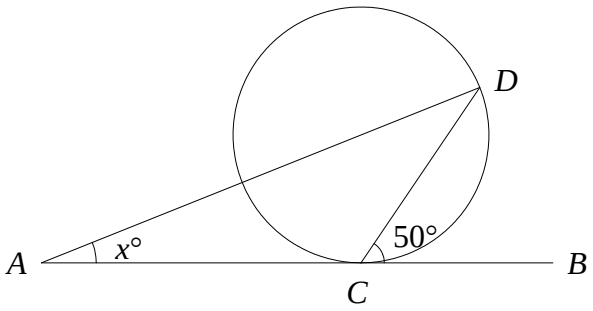
Questions 1 to 4 each carries 3 marks.

1. 小琪做家課時要計算一道形如 $a+b\times c$ 的算式（其中 a、b、c 是大於 1 的整數）。小琪忘了要先乘除後加減，得出的結果比正確答案大了 2011。求 c。

   In a homework problem Angel had to compute an expression of the form $a+b\times c$ where a, b, c are integers greater than 1. She forgot that multiplication and division should precede addition and subtraction and ended up with a result which is 2011 greater than the correct answer. Find c.

2. 圖中，直線 AB 切圓於 C。D 是圓周上的一點，且 DA 穿過圓心。若 $\angle DCB = 50°$ 而 $\angle DAB = x°$，求 x。

   In the figure, the straight line AB is tangent to the circle at C. D is a point on the circumference such that DA passes through the centre of the circle. If $\angle DCB = 50°$ and $\angle DAB = x°$, find x.



3. 顯恩把 20110402 的數字重新排列，得到一個不同的八位數，這個八位數跟原數相差 n。求 n 的最小可能值。

   Joseph rearranged the digits of 20110402 to form a different eight-digit number which differs from the original number by n. Find the smallest possible value of n.

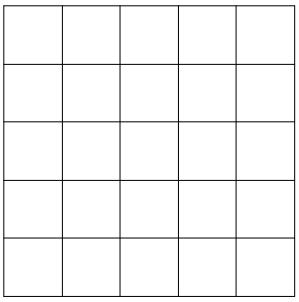
4. 嘉珮寫下了一個 20 項的正整數數列。從第二項起，每項都比之前的一項大 5。各項中最多有幾個是質數？

   Lyanne has written a sequence of 20 positive integers. Starting from the second term, each term is 5 greater than the previous term. What is the maximum number of terms that are prime?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 在一個遊戲中，參加者需在圖中的 5×5 方格陣的 25 個小格中每次選一個，直至選了五個中心成一直線的小格便算勝出。若一名參加者選了 n 個不同的小格後仍沒有勝出，求 n 的最大可能值。



In a game, the player shall choose one of the 25 small squares at a time in the 5×5 grid shown, and wins when five small squares whose centres lie on the same straight line have been chosen. If a player has not yet won after having chosen n different small squares, find the greatest possible value of n.

6. 某班有 5 名學生，學號分別爲 1 至 5。現每名學生可與其他同學成組，亦可自成一組，但規定兩名學號連續的學生不能同組。那麼共有多少種不同的分組方法？

In a class there are 5 students, numbered 1 to 5. A student can now form groups with others or form a group by him/herself, subject to the restriction that two students whose class numbers are consecutive cannot be in the same group. How many different ways of grouping are there?

7. 若實數 y 滿足 $y^y=10$，求 $y\log(\log y)$ 的值。

If the real number y satisfies $y^y=10$, find the value of $y\log(\log y)$.

8. 若 $1\le x\le10$ 和 $1\le y\le10$，求 $y^2x-3y(3-x)$ 的最小可能值。

If $1\le x\le10$ and $1\le y\le10$, find the smallest possible value of $y^2x-3y(3-x)$.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

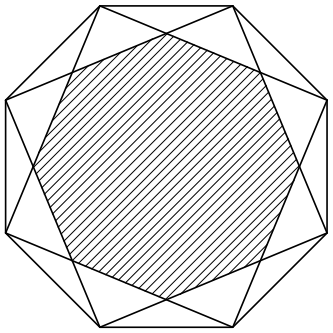
9. 若把 $5^{64}-1$ 寫成二進制，其末尾有多少個零？

How many ending zeros are there when $5^{64}-1$ is written in binary notation?

10. 對正整數 n，設 a_n 和 b_n 分別爲方程 $n(n+1)x^2-(2n+1)x+1=0$ 的兩個根（其中 $a_n>b_n$）。求 $(a_1+a_2+\cdots+a_{2011})-(b_1+b_2+\cdots+b_{2011})$ 的值。

For positive integer n, let a_n and b_n (where $a_n>b_n$) denote the two roots of the equation $n(n+1)x^2-(2n+1)x+1=0$. Find the value of $(a_1+a_2+\cdots+a_{2011})-(b_1+b_2+\cdots+b_{2011})$.

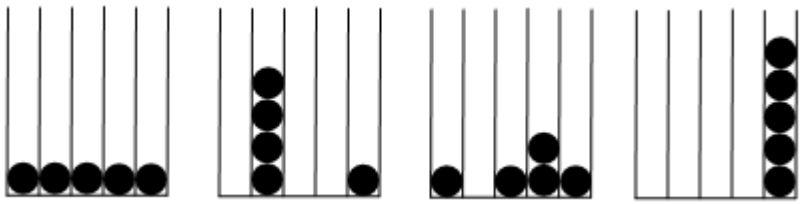
11. 若把一個邊長爲 1 的正八邊形的八個頂點梅花間竹地分成兩組，則每組的四個頂點均組成一個正方形。求兩個正方形重疊部分的面積。



If the eight vertices of a regular octagon of side length 1 are divided into two groups in an alternate manner, the four vertices in each group will form a square. Find the area of the region overlapped by the two squares.

12. 在一個遊戲中，參加者需把 5 個球射進 5 條垂直的坑道。如果 5 個球最終成一直線，參加者便勝出。（下圖顯示了四個例子，當中只有第一個和第四個勝出。）如果射了某數目的球後已經肯定無法勝出，那麼遊戲立即結束，參加者不能再射餘下的球。若每個球都必定會射進其中一條坑道，且射進各坑道的機會均等，求射了 3 球後遊戲便結束的概率。

In a game, the player has to shoot 5 balls into one of the 5 vertical tracks, and wins if the 5 balls turn out to be collinear. (The figure below shows four examples, among which only the first and fourth outcomes are winning.) If the player is certain not to win after shooting a certain number of balls, the game ends immediately without shooting the remaining balls. If each ball must get into one of the tracks with equal probability, find the probability that the game ends after 3 balls are shot.



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 設 m 和 n 爲正整數。若它們的最大公因數和它們的最小公倍數相差 2011，求 n 所有可能值之和。

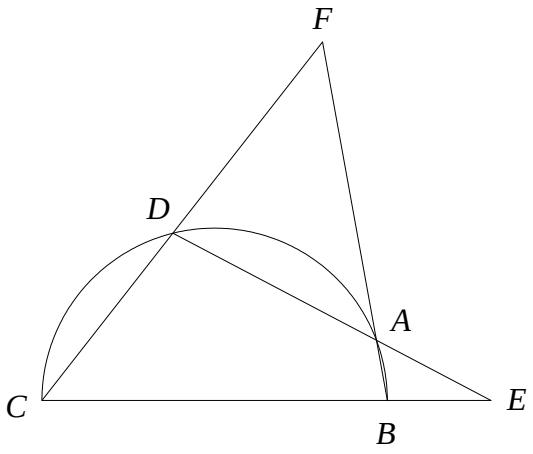
Let m and n be positive integers. If their H.C.F. and their L.C.M. differ by 2011, find the sum of all possible values of n.

14. 浩宏把水注進一個長方體至半滿，該長方體的底是一個面積 28 cm^2 的正方形。當他把長方體沿其中一條底邊傾側 $x°$ 時，水面的面積變成 35 cm^2；當他把長方體再傾側 $x°$ 後，水面的面積變成 n cm^2。求 n。（假設水面一直沒有觸及柱體底部。）

Adrian fills a cuboid to half-full. The base of the cuboid is a square with area 28 cm^2. When he tilts the cuboid for an angle of $x°$ along one of the edges of the base, the area of the water surface is 35 cm^2; when he further tilts the cuboid by another $x°$, the area of the water surface becomes n cm^2. Find n. (Assume that the water surface does not touch the base of the prism.)

15. 圖中，BC 是半圓的直徑，A、D 爲圓周上的兩點。CB 與 DA 延長後交於 E，CD 與 BA 延長後交於 F。若 $BC = 2$、$\angle CED = 15°$ 且 $\angle CFB = 45°$，求 AE^2，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示（其中 a、b 爲有理數）。

In the figure, BC is the diameter of the semi-circle and A, D are points on the circumference. CB and DA are extended to meet at E, while CD and BA are extended to meet at F. If $BC = 2$, $\angle CED = 15°$ and $\angle CFB = 45°$, find AE^2 in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where a, b are rational numbers.



16. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 PCIMC 所代表的五位數的最小可能值。

In the addition shown, each letter represents a different digit from 0 to 9. Find the smallest possible value of the five-digit number represented by PCIMC.

$$\begin{array}{rrrrrr} & T & E & N & T & H \\ + & P & C & I & M & C \\ \hline 1 & 1 & 0 & 4 & 0 & 2 \end{array}$$

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 某國家有 n 個城市，其中 $n > 20$。國王下令要興建一些道路，並須滿足以下條件：

- 每條道路連接兩個不同的城市。
- 任何 3 個城市均不能組成「孤立群」（即當中任何兩個城市均沒有道路連接）。
- 任何 20 個城市均不能組成「觀光圈」（即可以從 20 個城市的其中一個出發，經所建的道路到訪其餘 19 個城市，當中不會重複到訪城市也不會經過其他城市，最後返回起點）。

後來，該國的一名數學家證明了國王的要求是不可能達到的。求 n 的最小可能值。

There are n cities in a country, where $n > 20$. The king decided that some roads are to be built subject to the following requirements.

- Each road connects two different cities.
- Any 3 cities must not form an 'isolated group' (which means no two cities are connected by a road).
- Any 20 cities must not form a 'sightseeing loop' (which means one can start from one of the 20 cities and travel along the roads built to visit the other 19 cities without visiting the same city twice nor passing through other cities, and finally go back to the starting city).

It was later proved by a mathematician in the country that the requirements of the king cannot be fulfilled. Find the smallest possible value of n.

18. 求當 $C_2^{1234} + C_6^{1234} + C_{10}^{1234} + \cdots + C_{1230}^{1234} + C_{1234}^{1234}$ 除以 30 時的餘數。

Find the remainder when $C_2^{1234} + C_6^{1234} + C_{10}^{1234} + \cdots + C_{1230}^{1234} + C_{1234}^{1234}$ is divided by 30.

19. 一個袋子中有 1 張紅牌、2 張黃牌和 7 張白牌。小莉每次從袋子中抽出一張牌然後放回袋子，直至抽到紅牌或連續抽到兩張黃牌便停止。求她抽了第五張牌後停止的概率。

In a bag there is 1 red card, 2 yellow cards and 7 white cards. Each time Lily draws one card from the bag and then puts it back into the bag, until she draws a red card or two consecutive yellow cards, at which point she stops. Find the probability that she stops after drawing the fifth card.

20. 陳先生、李先生、張先生和黃先生分別是教師、警察、醫生和律師，而他們的妻子則分別是律師、教師、警察和醫生。這四對夫婦參加了一個象棋比賽，賽會以抽籤形式把八人分成二人一組對賽，方法如下：把八個分別寫上八人的名字的球放進一個袋子，然後逐一抽出，第一位被抽出的與第二位被抽出的對賽，第三位被抽出的與第四位被抽出的對賽，如此類推。惟出現以下任何一種情況時，則最近一個抽出的球無效，須放回袋子內重新抽出一個球：

(1) 抽中兩夫婦或兩名從事同一職業的人對賽

(2) 在餘下的抽籤過程中，肯定會出現兩夫婦或兩名從事同一職業的人對賽（例如：抽出第六個球後，袋子中只剩下代表陳先生和陳太太的球）

求抽籤過程中需要重新抽球的概率。

Mr Chan, Mr Lee, Mr Cheung and Mr Wong are a teacher, a policeman, a doctor and a lawyer respectively. Their wives are a lawyer, a teacher, a policeman and a doctor respectively. These four couples joined a chess competition. The organiser divided the eight participants into four groups of two to play against each other, as follows. Eight balls with the names of the eight contestants were put into a bag and then drawn one by one. The person first drawn would play against the one drawn second; the person drawn third would play against the one drawn fourth, and so on. However, if any of the following situations occurs, the latest ball drawn would be invalid and it would be put back into the bag and a ball had to be drawn again:

(1) The draw requires a couple or two people with the same occupation to play against each other.

(2) During the remaining draw, the situation where a couple or two people with the same occupation playing against each other would definitely occur (e.g. only the balls representing Mr Chan and Mrs Chan were left after the sixth ball was drawn).

Find the probability that a ball has to be redrawn during the process of the draw.

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