第十屆培正數學邀請賽

10th Pui Ching Invitational Mathematics Competition

決賽（中三組）

Final Event (Secondary 3)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確値表達，並化至最簡。不接受近似値。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 某數可寫成三個連續正整數之積，且可被 5 整除。該數的最小可能值是甚麼？

   A number can be written as the product of three consecutive positive integers and is divisible by 5. What is the smallest possible value of the number?

2. 在一個排球比賽中，開始時的計分牌顯示 $0:0$，每球勝出的一方可得 1 分，先得到 25 分的一方勝出（不設「刁時」，即如果打成 $24:24$ 平手，再勝出一球的一方便會勝出）。計分牌顯示的分數有多少個不同的可能？（註：若 $a \neq b$，則 $a:b$ 和 $b:a$ 視爲不同的分數。）

   In a volleyball match, the scoreboard initially shows $0:0$. Each time the winner scores 1 point, and whichever side gets 25 points first wins (no 'deuce' is played, i.e. when the score is $24:24$, the side getting the next point wins). How many different scores are possible on the scoreboard? (Note: We regard $a:b$ and $b:a$ to be different scores if $a \neq b$.)

3. 現有兩個容量分別爲 3 公升和 5 公升的瓶子和一條水喉。開始時，兩個瓶子都是空的。在每個步驟中，我們可以選擇進行以下操作的其中一項：

   - 利用水喉把一個空的瓶子或一個未盛滿水的瓶子注滿。
   - 把一個瓶子中的水倒掉。
   - 把一個瓶子中的水倒到另一個瓶子，直至所有水已倒進另一瓶子或另一瓶子已滿。

   若要使得 5 公升的瓶子內盛有剛好 4 公升水，問最少要經過多少個步驟？

   There are two bottles with capacities 3 litres and 5 litres respectively, as well as a tap. In the beginning, both bottles are empty. In each step, we can perform one of the following operations:

   - Fill an empty or a partially-filled bottle full of water using the water tap.
   - Empty a bottle.
   - Pour the water in one bottle into another bottle, until the other bottle is full or all water has been poured into the other bottle.

   What is the minimum number of steps needed in order to have exactly 4 litres of water in the 5-litre bottle?

4. 顯恩把 20110402 的數字重新排列，得到一個不同的八位數，這個八位數跟原數相差 n。求 n 的最小可能值。

Joseph rearranged the digits of 20110402 to form a different eight-digit number which differs from the original number by n. Find the smallest possible value of n.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 某扇形的半徑是 2，周界是 8。求它的面積。

A sector has radius 2 and perimeter 8. Find its area.

6. 如果一個 12 小時時鐘的鐘面上沒有數字，那麼當我們看著時鐘的鏡像時，會看出另一個時間，例如：在 2 時看時鐘的鏡像會看出 10 時，兩者相差了 4 小時（這裡相差 4 小時是指最接近的 10 時和 2 時，例如同一天的上午 10 時和下午 2 時便相差 4 小時，下同）。在一天中，實際時間和鏡像時間相差剛好 2.5 小時的情況會出現多少次？

If there are no numerical labels on a 12-hour clock, we will see another time by looking at the mirror image of the clock. For instance, at 2 o'clock the mirror image will look like 10 o'clock. The difference between the two times is 4 hours (referring to the interval between the closest 10 o'clock and 2 o'clock, for instance 10 am and 2 pm of the same day, same for below). In a day, how many instances are there such that the difference between the actual time and the time in the mirror image is exactly 2.5 hours?

7. 設 n 為 2011 位數 999…99。在 n^4 的數字中，有多少個是 9？

Let n be the 2011-digit number 999…99. How many 9's are there in the digits of n^4?

8. 設 A 是 18 位數、B 是 24 位數。若先寫下 A 再寫下 B，便可得到一個 42 位數 C，當中有 k 個不同的數字。若先寫下 B 再寫下 A，所得的 42 位數亦是 C。求 k 所有可能值之和。

Let A be a 18-digit number and B be a 24-digit number. If one first writes A and then writes B, one gets a 42-digit number C, which consists of k different digits. It turns out that if one first writes B and then writes A, one also gets the 42-digit number C. Find the sum of all possible values of k.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 若一個正整數可寫成 k^2+k（其中 k 是正整數）的形式，則我們稱它爲「好數」。例如：因爲 $6^2+6=42$，故此 42 是「好數」。在首 2011 個正整數中，有多少個可寫成兩個「好數」之差？

   A positive integer is said to be ‘good’ if it can be expressed in the form k^2+k for some positive integer k. For instance, since $6^2+6=42$, we say that 42 is ‘good’. How many of the first 2011 positive integers can be expressed as the difference between two ‘good’ numbers?

10. 某次測驗有三名學生小文、小華和小強參加。測驗共設 n 題，分別佔 1、2、⋯、n 分，每名考生在每題均取得 0 分或該題的全部分數。已知

   - 小文答對了連續的 3 題，其他題目答錯；
   - 小華答對了連續的 4 題，其他題目答錯；
   - 小強答對了連續的 7 題，其他題目答錯；
   - 小文、小華和小強的分數相同；
   - 在其中一題，三人當中剛好有兩人答對；且
   - 在最後一題，三人當中剛好有一人答對。

   求 n。

   Three students, Raymond, Steven and Tony, took a test. In the test there were n questions, worth 1, 2, …, n marks respectively. In each question a candidate either got zero or all the marks allocated. Suppose that

   - Raymond answered three consecutive questions correctly and the rest wrongly;
   - Steven answered four consecutive questions correctly and the rest wrongly;
   - Tony answered seven consecutive questions correctly and the rest wrongly;
   - Raymond, Steven and Tony got the same marks;
   - a question was correctly answered by exactly two of the three students; and
   - the last question was correctly answered by exactly one of the three students.

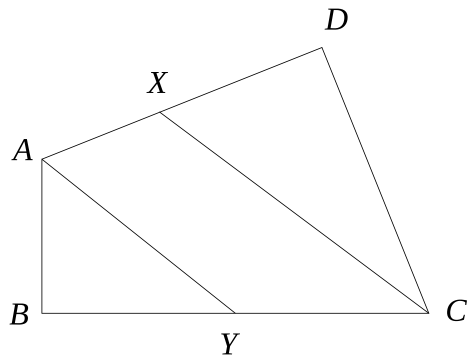
   Find n.

11. 2011 個連續正奇數之和是個立方數。若 k 是當中最小的一個奇數，求 k 的最小可能值。

The sum of 2011 consecutive odd positive integers is a cubic number. If k is the smallest of these odd numbers, find the smallest possible value of k.

12. 在四邊形 $ABCD$ 中， $AB=4$ 、 $CD=6$ ，且 B 和 D 都是直角。X 和 Y 分別是 AD 和 BC 上的點，使得 AX 和 CY 的長度均爲整數。若四邊形 $AXCY$ 的面積是 2011，則 AX 的長度有多少個不同的可能值？

In quadrilateral $ABCD$, $AB=4$, $CD=6$ and both B and D are right angles. X and Y are points on AD and BC respectively such that the lengths of AX and CY are both integers. If quadrilateral $AXCY$ has area 2011, how many different possible values are there for the length of AX?

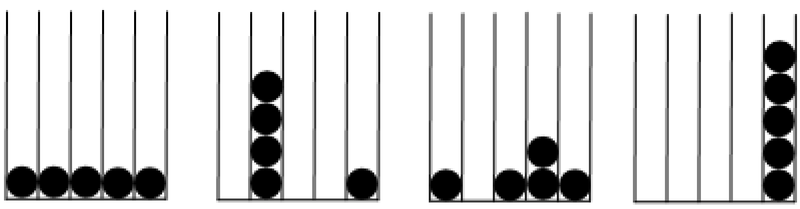


第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 在一個遊戲中，參加者需把 5 個球射進 5 條垂直的坑道。如果 5 個球最終成一直線，參加者便勝出。（下圖顯示了四個例子，當中只有第一個和第四個勝出。）如果射了某數目的球後已經肯定無法勝出，那麼遊戲立即結束，參加者不能再射餘下的球。若每個球都必定會射進其中一條坑道，且射進各坑道的機會均等，求遊戲結束時射了的球的總數期望值。

In a game, the player has to shoot 5 balls into one of the 5 vertical tracks, and wins if the 5 balls turn out to be collinear. (The figure below shows four examples, among which only the first and fourth outcomes are winning.) If the player is certain not to win after shooting a certain number of balls, the game ends immediately without shooting the remaining balls. If each ball must get into one of the tracks with equal probability, find the expected value of the number of balls shot when the game ends.



14. 某三角形的三條高分別長 2011、402 和 n，其中 n 是正整數。問 n 有多少個不同的可能值？

The three altitudes of a triangle have lengths 2011, 402 and n, where n is a positive integer. How many different possible values of n are there?

15. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 PCIMC 所代表的五位數的最小可能值。

In the addition shown, each letter represents a different digit from 0 to 9. Find the smallest possible value of the five-digit number represented by PCIMC.

$$\begin{array}{r} \text{T E N T H} \\ +\ \text{P C I M C} \\ \hline 1\ 1\ 0\ 4\ 0\ 2 \end{array}$$

16. 設 $a_1 = 40$，並對正整數 n 定義

$$a_{n+1} = \begin{cases} a_n - \frac{1}{n} & \text{若 } a_n^2 < 2011 \\ a_n + \frac{1}{n} & \text{若 } a_n^2 > 2011 \\ 2011 & \text{若 } a_n^2 = 2011 \end{cases}$$

求 $\lim_{n\to\infty} a_n$。（換句話說，當 n 很大時，a_n 會趨近甚麼數？）

Let $a_1 = 40$. For positive integer n, we define

$$a_{n+1} = \begin{cases} a_n - \frac{1}{n} & \text{if } a_n^2 < 2011 \\ a_n + \frac{1}{n} & \text{if } a_n^2 > 2011 \\ 2011 & \text{if } a_n^2 = 2011 \end{cases}$$

Find $\lim_{n\to\infty} a_n$. (In other words, to what number will a_n approach when n is large?)

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在一個遊戲中，11 人圍圈而坐，另外有 11 張卡紙，其中 8 張是空白的，另外 3 張則分別寫上「A」、「B」和「C」。開始時，每人隨機獲發一張卡紙。在每個回合中，每人都會把手上的卡紙傳給右方的人，然後主持人會要求知道「A」、「B」和「C」三張卡紙分別在誰人手上者舉手。結果在首三個回合中均沒有人舉手，而在第四回合中則有一人舉手。若第五和第六回合分別有 x 人和 y 人舉手，求乘積 xy。（各人都不會看到別人手上的卡紙，而且各人都是聰明的，即有足夠資料時便能作出推論。）

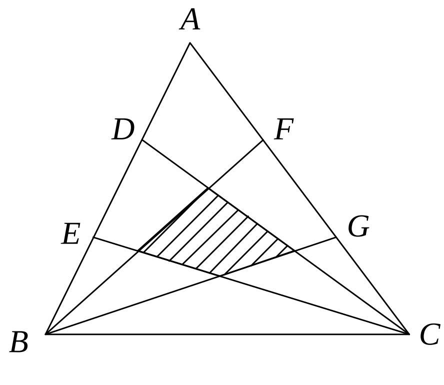
In a game, 11 people sat in a circle. There were 11 cards, 8 of which were blank, and the numbers ‘A’, ‘B’ and ‘C’ were written on the other 3 respectively. One card was distributed to each person at random at the beginning. In each round, each person passed the card to the person on his right, and then the players would be asked to raise his hands if he could tell which people were holding the cards with ‘A’, ‘B’ and ‘C’ written respectively. It turned out that nobody raised hands during the first three rounds, while one person raised hands during the fourth round. During the fifth and sixth rounds, there were x and y people raising hands respectively. Find the product xy. (The people could not see the cards held by others. They are also intelligent, so that deductions can be made whenever sufficient information is available.)

18. 若把 $\frac{729^3+1^3}{729^3+728^3}$ 、 $\frac{729^3+2^3}{729^3+727^3}$ 、…、 $\frac{729^3+728^3}{729^3+1^3}$ 分別寫成最簡分數，求所有分子之和。

When each of the numbers $\frac{729^3+1^3}{729^3+728^3}$, $\frac{729^3+2^3}{729^3+727^3}$, …, $\frac{729^3+728^3}{729^3+1^3}$ is written as a fraction in the lowest term, find the sum of all the numerators.

19. 圖中，ABC 是面積爲 1 的三角形。D、E 是 AB 上的點，F、G 則是 AC 上的點，使得 $AD = DE = EB$ 和 $AF = FG = GC$。求 BF、BG、CD 和 CE 圍成的區域的面積。

In the figure, ABC is a triangle with area 1. D, E are points on AB while F, G are points on AC such that $AD = DE = EB$ and $AF = FG = GC$. Find the area of the region bounded by BF, BG, CD and CE.



20. 陳先生、李先生、張先生和黃先生分別是教師、警察、醫生和律師，而他們的妻子則分別是律師、教師、警察和醫生。這四對夫婦參加了一個象棋比賽，賽會以抽籤形式把八人分成二人一組對賽，方法如下：把八個分別寫上八人的名字的球放進一個袋子，然後逐一抽出，第一位被抽出的與第二位被抽出的對賽，第三位被抽出的與第四位被抽出的對賽，如此類推。惟出現以下任何一種情況時，則最近一個抽出的球無效，須放回袋子內重新抽出一個球：

(1) 抽中兩夫婦或兩名從事同一職業的人對賽

(2) 在餘下的抽籤過程中，肯定會出現兩夫婦或兩名從事同一職業的人對賽（例如：抽出第六個球後，袋子中只剩下代表陳先生和陳太太的球）

求抽籤過程中需要重新抽球的概率。

Mr Chan, Mr Lee, Mr Cheung and Mr Wong are a teacher, a policeman, a doctor and a lawyer respectively. Their wives are a lawyer, a teacher, a policeman and a doctor respectively. These four couples joined a chess competition. The organiser divided the eight participants into four groups of two to play against each other, as follows. Eight balls with the names of the eight contestants were put into a bag and then drawn one by one. The person first drawn would play against the one drawn second; the person drawn third would play against the one drawn fourth, and so on. However, if any of the following situations occurs, the latest ball drawn would be invalid and it would be put back into the bag and a ball had to be drawn again:

(1) The draw requires a couple or two people with the same occupation to play against each other.

(2) During the remaining draw, the situation where a couple or two people with the same occupation playing against each other would definitely occur (e.g. only the balls representing Mr Chan and Mrs Chan were left after the sixth ball was drawn).

Find the probability that a ball has to be redrawn during the process of the draw.

全卷完

END OF PAPER