第十屆培正數學邀請賽

10th Pui Ching Invitational Mathematics Competition

決賽（中二組）

Final Event (Secondary 2)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 小琪做家課時要計算一道形如 $a+b\times c$ 的算式（其中 a、b、c 是大於 1 的整數）。小琪忘了要先乘除後加減，得出的結果比正確答案大了 2011。求 c。

   In a homework problem Angel had to compute an expression of the form $a+b\times c$ where a, b, c are integers greater than 1. She forgot that multiplication and division should precede addition and subtraction and ended up with a result which is 2011 greater than the correct answer. Find c.

2. 某數可寫成三個連續正整數之積，且可被 5 整除。該數的最小可能值是甚麼？

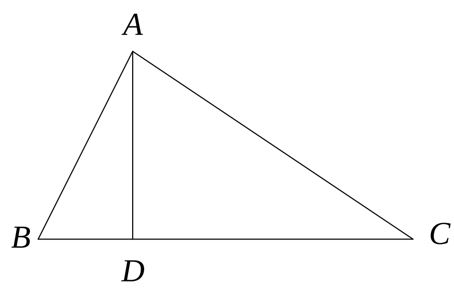
   A number can be written as the product of three consecutive positive integers and is divisible by 5. What is the smallest possible value of the number?

3. 某 n 位數的數字之和是 2011。求 n 的最小可能值。

   The sum of digits of an n-digit number is 2011. Find the smallest possible value of n.

4. 在 ΔABC 中，D 是 A 到 BC 的垂足。若 $AB:AC=1:2$，$DB=3$ 且 $DC=9$，求 DA。

   In ΔABC, D is the foot of the perpendicular from A to BC. If $AB:AC=1:2$, $DB=3$ and $DC=9$, find DA.



第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. ABC 是個三角形，它的面積是 12。P 是三角形內的一點。已知 X、Y 和 Z 分別是 P 繞 A、B 和 C 點旋轉 180° 後的影像。求 ΔXYZ 的面積。

   ABC is a triangle of area 12. P is a point in the triangle. It is known that X, Y and Z are the images of P when rotated about A, B and C by 180° respectively. Find the area of ΔXYZ.

6. 某班有 5 名學生，學號分別爲 1 至 5。現每名學生可與其他同學成組，亦可自成一組，但規定兩名學號連續的學生不能同組。那麼共有多少種不同的分組方法？

In a class there are 5 students, numbered 1 to 5. A student can now form groups with others or form a group by him/herself, subject to the restriction that two students whose class numbers are consecutive cannot be in the same group. How many different ways of grouping are there?

7. 已知 a、b、c 和 d 都是正整數，其中 $a>b>c>d$。若 $a-b$、$a-c$、$a-d$、$b-c$、$b-d$ 和 $c-d$ 這六個數當中剛好有五個是質數，求 a 的最小可能值。

It is known that a, b, c and d are positive integers where $a>b>c>d$. If exactly five of the six integers $a-b$, $a-c$, $a-d$, $b-c$, $b-d$ and $c-d$ are prime, find the smallest possible value of a.

8. 設 n 爲 2011 位數 999…99。在 n^4 的數字中，有多少個是 9？

Let n be the 2011-digit number 999…99. How many 9's are there in the digits of n^4?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 若一個正整數可寫成 k^2+k（其中 k 是正整數）的形式，則我們稱它爲「好數」。例如：因爲 $6^2+6=42$，故此 42 是「好數」。在首 2011 個正整數中，有多少個可寫成兩個「好數」之差？

A positive integer is said to be 'good' if it can be expressed in the form k^2+k for some positive integer k. For instance, since $6^2+6=42$, we say that 42 is 'good'. How many of the first 2011 positive integers can be expressed as the difference between two 'good' numbers?

10. 設 x_1、x_2、⋯、x_{100} 爲介乎 0.5 和 0.75 之間的實數（包括 0.5 和 0.75）。求 $x_1(1-x_2)+x_2(1-x_3)+x_3(1-x_4)+\cdots+x_{99}(1-x_{100})+x_{100}(1-x_1)$ 的最大可能值。

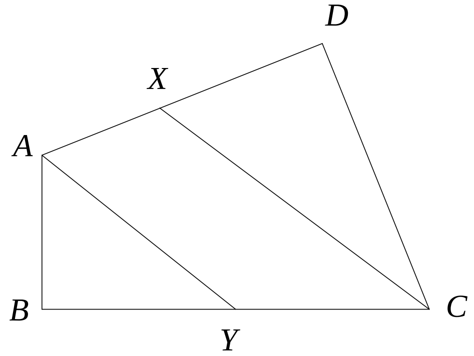
Let x_1, x_2, …, x_{100} be real numbers between 0.5 and 0.75 inclusive. Find the greatest possible value of $x_1(1-x_2)+x_2(1-x_3)+x_3(1-x_4)+\cdots+x_{99}(1-x_{100})+x_{100}(1-x_1)$.

11. 若把 $5^{64}-1$ 寫成二進制，其末尾有多少個零？

How many ending zeros are there when $5^{64}-1$ is written in binary notation?

12. 在四邊形 *ABCD* 中， $AB=4$ 、 $CD=6$ ，且 *B* 和 *D* 都是直角。*X* 和 *Y* 分別是 *AD* 和 *BC* 上的點，使得 *AX* 和 *CY* 的長度均爲整數。若四邊形 *AXCY* 的面積是 2011，則 *AX* 的長度有多少個不同的可能值？

In quadrilateral *ABCD*, $AB=4$, $CD=6$ and both *B* and *D* are right angles. *X* and *Y* are points on *AD* and *BC* respectively such that the lengths of *AX* and *CY* are both integers. If quadrilateral *AXCY* has area 2011, how many different possible values are there for the length of *AX*?



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 某三角形的三條高分別長 2011、402 和 *n*，其中 *n* 是正整數。問 *n* 有多少個不同的可能值？

The three altitudes of a triangle have lengths 2011, 402 and *n*, where *n* is a positive integer. How many different possible values of *n* are there?

14. 設 $a_1=40$ ，並對正整數 *n* 定義

$$a_{n+1}=\begin{cases} a_n-\frac{1}{n} & \text{若 } a_n{}^2<2011 \\ a_n+\frac{1}{n} & \text{若 } a_n{}^2>2011 \\ 2011 & \text{若 } a_n{}^2=2011 \end{cases}$$

求 $\lim_{n\to\infty} a_n$ 。（換句話說，當 *n* 很大時， a_n 會趨近甚麼數？）

Let $a_1=40$. For positive integer *n*, we define

$$a_{n+1}=\begin{cases} a_n-\frac{1}{n} & \text{if } a_n{}^2<2011 \\ a_n+\frac{1}{n} & \text{if } a_n{}^2>2011 \\ 2011 & \text{if } a_n{}^2=2011 \end{cases}$$

Find $\lim_{n\to\infty} a_n$. (In other words, to what number will a_n approach when *n* is large?)

15. 在一個排球比賽中，開始時的計分牌顯示 $0:0$，每球勝出的一方可得 1 分，先得到 25 分的一方勝出（不設「刁時」，即如果打成 $24:24$ 平手，再勝出一球的一方便會勝出）。小權觀看了 n 場比賽後，發現所有可能的分數都已經在計分牌上出現。求 n 的最小可能值。（註：若 $a \neq b$，則 $a:b$ 和 $b:a$ 視爲不同的分數。）

In a volleyball match, the scoreboard initially shows $0:0$. Each time the winner scores 1 point, and whichever side gets 25 points first wins (no 'deuce' is played, i.e. when the score is $24:24$, the side getting the next point wins). After watching n matches, Donald found that all possible scores have already appeared on the scoreboard. Find the smallest possible value of n. (Note: We regard $a:b$ and $b:a$ to be different scores if $a \neq b$.)

16. 求滿足以下條件的五位正整數的數目：

- 每個數字皆不是 0。
- 該五位數可被 5 整除。
- 若把最後一位數字刪去，所得的四位數可被 4 整除。
- 若把最後兩位數字刪去，所得的三位數可被 3 整除。
- 若把最後三位數字刪去，所得的兩位數可被 2 整除。

Find the number of 5-digit positive integers satisfying the following conditions:

- Every digit is non-zero.
- The 5-digit number is divisible by 5.
- If the last digit is removed, the resulting 4-digit number is divisible by 4.
- If the last two digits are removed, the resulting 3-digit number is divisible by 3.
- If the last three digits are removed, the resulting 2-digit number is divisible by 2.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 PCIMC 所代表的五位數的最大可能值。

In the addition shown, each letter represents a different digit from 0 to 9. Find the greatest possible value of the five-digit number represented by PCIMC.

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   T E N T H
+  P C I M C
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 1 1 0 4 0 2
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18. 在一個遊戲中，11 人圍圈而坐，另外有 11 張卡紙，其中 8 張是空白的，另外 3 張則分別寫上「A」、「B」和「C」。開始時，每人隨機獲發一張卡紙。在每個回合中，每人都會把手上的卡紙傳給右方的人，然後主持人會要求知道「A」、「B」和「C」三張卡紙分別在誰人手上者舉手。結果在首三個回合中均沒有人舉手，而在第四回合中則有一人舉手。若第五和第六回合分別有 x 人和 y 人舉手，求乘積 xy。（各人都不會看到別人手上的卡紙，而且各人都是聰明的，即有足夠資料時便能作出推論。）

In a game, 11 people sat in a circle. There were 11 cards, 8 of which were blank, and the numbers ‘A’, ‘B’ and ‘C’ were written on the other 3 respectively. One card was distributed to each person at random at the beginning. In each round, each person passed the card to the person on his right, and then the players would be asked to raise his hands if he could tell which people were holding the cards with ‘A’, ‘B’ and ‘C’ written respectively. It turned out that nobody raised hands during the first three rounds, while one person raised hands during the fourth round. During the fifth and sixth rounds, there were x and y people raising hands respectively. Find the product xy. (The people could not see the cards held by others. They are also intelligent, so that deductions can be made whenever sufficient information is available.)

19. 在一個重組句子遊戲中，參加者需要把 6 張分別寫上「數」、「學」、「很」、「有」、「趣」和「味」的卡片重新排列。遊戲的計分方法如下：正確的排列是「數學很有趣味」，在重新排列卡片後，我們把 6 張排列後的卡片分成最長的連續正確段（即連續地在正確排列中出現的卡片），每個由 k 張卡片組成的最長連續正確段可得 2^k 分。例如：如果排列是「味有趣數學很」，則最長的連續正確段分別是「味」、「有趣」和「數學很」，因此得分是 $2^1+2^2+2^3=14$。若把卡片隨意排列，則最有可能得到的分數是 S，且有 n 個不同的排列可以得到 S 分。求 n。

In a sentence reconstruction game, players had to rearrange 6 cards labelled ‘mathematics’, ‘is’, ‘a’, ‘very’, ‘interesting’ and ‘subject’. The score is computed as follows. With ‘mathematics is a very interesting subject’ being the correct order, the 6 rearranged cards are divided into maximal consecutive correct segments (i.e. consecutive cards which appear in the correct order). Each maximal consecutive correct segment consisting of k cards is worth 2^k points. For instance, for the rearrangement ‘subject very interesting mathematics is a’, the maximal consecutive correct segments are ‘subject’, ‘very interesting’ and ‘mathematics is a’, and hence the score is $2^1+2^2+2^3=14$. By a random rearrangement of the cards, the most probable score is S with n different rearrangements leading to such score. Find n.

20. 若把 $\frac{729^3+1^3}{729^3+728^3}$ 、 $\frac{729^3+2^3}{729^3+727^3}$ 、…、 $\frac{729^3+728^3}{729^3+1^3}$ 分別寫成最簡分數，求所有分子之和。

When each of the numbers $\frac{729^3+1^3}{729^3+728^3}$, $\frac{729^3+2^3}{729^3+727^3}$, …, $\frac{729^3+728^3}{729^3+1^3}$ is written as a fraction in the lowest term, find the sum of all the numerators.

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END OF PAPER