第十屆培正數學邀請賽

10th Pui Ching Invitational Mathematics Competition

決賽（中一組）

Final Event (Secondary 1)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 某 n 位數的數字之和是 2011。求 n 的最小可能値。

   The sum of digits of an n-digit number is 2011. Find the smallest possible value of n.

2. 在一個平行四邊形中，其中一隻內角是另一隻內角的 3 倍。若最大的一隻內角是 $x°$，求 x。

   In a parallelogram, one interior angle is 3 times another. If the largest interior angle is $x°$, find x.

3. 在首 2011 個正整數中，有多少個的數字之和是 5 的倍數？

   How many of the first 2011 positive integers have their sum of digits being a multiple of 5?

4. 現有兩個容量分別爲 3 公升和 5 公升的瓶子和一條水喉。開始時，兩個瓶子都是空的。在每個步驟中，我們可以選擇進行以下操作的其中一項：

   - 利用水喉把一個空的瓶子或一個未盛滿水的瓶子注滿。
   - 把一個瓶子中的水倒掉。
   - 把一個瓶子中的水倒到另一個瓶子，直至所有水已倒進另一瓶子或另一瓶子已滿。

   若要使得 5 公升的瓶子內盛有剛好 4 公升水，問最少要經過多少個步驟？

   There are two bottles with capacities 3 litres and 5 litres respectively, as well as a tap. In the beginning, both bottles are empty. In each step, we can perform one of the following operations:

   - Fill an empty or a partially-filled bottle full of water using the water tap.
   - Empty a bottle.
   - Pour the water in one bottle into another bottle, until the other bottle is full or all water has been poured into the other bottle.

   What is the minimum number of steps needed in order to have exactly 4 litres of water in the 5-litre bottle?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. *ABC* 是個三角形，它的面積是 12。*P* 是三角形內的一點。已知 *X*、*Y* 和 *Z* 分別是 *P* 繞 *A*、*B* 和 *C* 點旋轉 180° 後的影像。求 Δ*XYZ* 的面積。

   ABC is a triangle of area 12. *P* is a point in the triangle. It is known that *X*, *Y* and *Z* are the images of *P* when rotated about *A*, *B* and *C* by 180° respectively. Find the area of Δ*XYZ*.

6. 如果一個 12 小時時鐘的鐘面上沒有數字，那麼當我們看著時鐘的鏡像時，會看出另一個時間，例如：在 2 時看時鐘的鏡像會看出 10 時，兩者相差了 4 小時（這裡相差 4 小時是指最接近的 10 時和 2 時，例如同一天的上午 10 時和下午 2 時便相差 4 小時，下同）。在一天中，實際時間和鏡像時間相差剛好 2.5 小時的情況會出現多少次？

   If there are no numerical labels on a 12-hour clock, we will see another time by looking at the mirror image of the clock. For instance, at 2 o'clock the mirror image will look like 10 o'clock. The difference between the two times is 4 hours (referring to the interval between the closest 10 o'clock and 2 o'clock, for instance 10 am and 2 pm of the same day, same for below). In a day, how many instances are there such that the difference between the actual time and the time in the mirror image is exactly 2.5 hours?

7. 已知 a、b、c 和 d 都是正整數，其中 $a>b>c>d$。若 $a-b$、$a-c$、$a-d$、$b-c$、$b-d$ 和 $c-d$ 這六個數當中剛好有五個是質數，求 a 的最小可能值。

   It is known that a, b, c and d are positive integers where $a>b>c>d$. If exactly five of the six integers $a-b$, $a-c$, $a-d$, $b-c$, $b-d$ and $c-d$ are prime, find the smallest possible value of a.

8. 設 *A* 是 18 位數、*B* 是 24 位數。若先寫下 *A* 再寫下 *B*，便可得到一個 42 位數 *C*，當中有 *k* 個不同的數字。若先寫下 *B* 再寫下 *A*，所得的 42 位數亦是 *C*。求 *k* 所有可能值之和。

   Let *A* be a 18-digit number and *B* be a 24-digit number. If one first writes *A* and then writes *B*, one gets a 42-digit number *C*, which consists of *k* different digits. It turns out that if one first writes *B* and then writes *A*, one also gets the 42-digit number *C*. Find the sum of all possible values of *k*.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 設 x_1 、 x_2 、…、 x_{100} 爲介乎 0.5 和 0.75 之間的實數（包括 0.5 和 0.75）。求 $x_1(1-x_2)+x_2(1-x_3)+x_3(1-x_4)+\cdots+x_{99}(1-x_{100})+x_{100}(1-x_1)$ 的最大可能值。

   Let x_1, x_2, …, x_{100} be real numbers between 0.5 and 0.75 inclusive. Find the greatest possible value of $x_1(1-x_2)+x_2(1-x_3)+x_3(1-x_4)+\cdots+x_{99}(1-x_{100})+x_{100}(1-x_1)$.

10. 某次測驗有三名學生小文、小華和小強參加。測驗共設 n 題，分別佔 1、2、…、n 分，每名考生在每題均取得 0 分或該題的全部分數。已知

    - 小文答對了連續的 3 題，其他題目答錯；
    - 小華答對了連續的 4 題，其他題目答錯；
    - 小強答對了連續的 7 題，其他題目答錯；
    - 小文、小華和小強的分數相同；
    - 在其中一題，三人當中剛好有兩人答對；且
    - 在最後一題，三人當中剛好有一人答對。

    求 n。

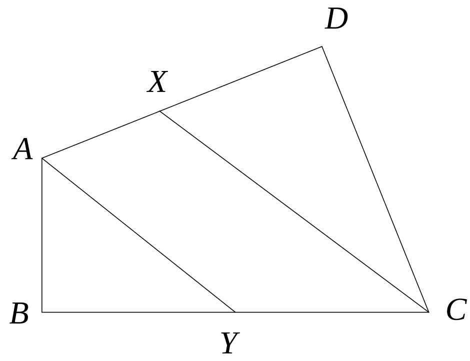
    Three students, Raymond, Steven and Tony, took a test. In the test there were n questions, worth 1, 2, …, n marks respectively. In each question a candidate either got zero or all the marks allocated. Suppose that

    - Raymond answered three consecutive questions correctly and the rest wrongly;
    - Steven answered four consecutive questions correctly and the rest wrongly;
    - Tony answered seven consecutive questions correctly and the rest wrongly;
    - Raymond, Steven and Tony got the same marks;
    - a question was correctly answered by exactly two of the three students; and
    - the last question was correctly answered by exactly one of the three students.

    Find n.

11. 在四邊形 $ABCD$ 中， $AB=4$ 、 $CD=6$ ，且 B 和 D 都是直角。X 和 Y 分別是 AD 和 BC 上的點，使得 AX 和 CY 的長度均爲整數。若四邊形 $AXCY$ 的面積是 2011，則 AX 的長度有多少個不同的可能值？

In quadrilateral *ABCD*, $AB=4$, $CD=6$ and both B and D are right angles. X and Y are points on AD and BC respectively such that the lengths of AX and CY are both integers. If quadrilateral *AXCY* has area 2011, how many different possible values are there for the length of AX?



12. 在一個重組句子遊戲中，參加者需要把 6 張分別寫上「數」、「學」、「很」、「有」、「趣」和「味」的卡片重新排列。遊戲的計分方法如下：正確的排列是「數學很有趣味」，在重新排列卡片後，我們把 6 張排列後的卡片分成最長的連續正確段（即連續地在正確排列中出現的卡片），每個由 k 張卡片組成的最長連續正確段可得 2^k 分。例如：如果排列是「味有趣數學很」，則最長的連續正確段分別是「味」、「有趣」和「數學很」，因此得分是 $2^1+2^2+2^3=14$。若把卡片隨意排列，則最有可能得到的分數是甚麼？

In a sentence reconstruction game, players had to rearrange 6 cards labelled ‘mathematics’, ‘is’, ‘a’, ‘very’, ‘interesting’ and ‘subject’. The score is computed as follows. With ‘mathematics is a very interesting subject’ being the correct order, the 6 rearranged cards are divided into maximal consecutive correct segments (i.e. consecutive cards which appear in the correct order). Each maximal consecutive correct segment consisting of k cards is worth 2^k points. For instance, for the rearrangement ‘subject very interesting mathematics is a’, the maximal consecutive correct segments are ‘subject’, ‘very interesting’ and ‘mathematics is a’, and hence the score is $2^1+2^2+2^3=14$. By a random rearrangement of the cards, what is the most probable score?

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 有多少個三位數 $\overline{abc}$ 的每個數字皆非零，且使得兩位數 $\overline{ab}$ 、$\overline{bc}$ 和 $\overline{ca}$ 皆可被 7 整除？

    How many three-digit numbers $\overline{abc}$ have the property that every digit is non-zero, and that the two-digit numbers $\overline{ab}$, $\overline{bc}$ and $\overline{ca}$ are all divisible by 7?

14. 設 m 和 n 爲正整數。若它們的最大公因數和它們的最小公倍數相差 2011，求 n 所有可能值之和。

    Let m and n be positive integers. If their H.C.F. and their L.C.M. differ by 2011, find the sum of all possible values of n.

15. 在一個排球比賽中，開始時的計分牌顯示 $0:0$，每球勝出的一方可得 1 分，先得到 25 分的一方勝出（不設「刁時」，即如果打成 $24:24$ 平手，再勝出一球的一方便會勝出）。小權觀看了 n 場比賽後，發現所有可能的分數都已經在計分牌上出現。求 n 的最小可能值。（註：若 $a \neq b$，則 $a:b$ 和 $b:a$ 視爲不同的分數。）

    In a volleyball match, the scoreboard initially shows $0:0$. Each time the winner scores 1 point, and whichever side gets 25 points first wins (no 'deuce' is played, i.e. when the score is $24:24$, the side getting the next point wins). After watching n matches, Donald found that all possible scores have already appeared on the scoreboard. Find the smallest possible value of n. (Note: We regard $a:b$ and $b:a$ to be different scores if $a \neq b$.)

16. 如果我們把一個正整數 n 的正因數從小至大排列然後逐一加起來，則可以從每次所得的和組成一個數列，例如：6 的正因數有 1、2、3、6，因此所得的數列是 (1, 3, 6, 12)。同樣，如果取 $n=1$ 和 $n=9$，所得的數列分別是 (1) 和 (1, 4, 13)。如果 n 自己有在數列中出現，則 n 稱爲「好數」。從以上例子可見，1 和 6 是「好數」，9 則不是「好數」。求所有小於 100 的「好數」之和。

    If we list the positive factors of a positive integer n in ascending order and then add them up one by one, then we get a sequence by recording the sum in each step. For instance, since 6 has positive factors 1, 2, 3 and 6, the sequence obtained will be (1, 2, 3, 6). Similarly, if we take $n=1$ and $n=9$, the sequence obtained will be (1) and (1, 4, 13) respectively. If the number n itself appears in the sequence, then it is said to be 'good'. The above examples thus show that 1 and 6 are 'good' while 9 is not. Find the sum of all 'good' numbers less than 100.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 PCIMC 所代表的五位數的最大可能值。

In the addition shown, each letter represents a different digit from 0 to 9. Find the greatest possible value of the five-digit number represented by PCIMC.

$$\begin{array}{r} T\ E\ N\ T\ H \\ +\quad P\ C\ I\ M\ C \\ \hline 1\ 1\ 0\ 4\ 0\ 2 \end{array}$$

18. 某國家有 n 個城市，其中 $n > 20$ 。國王下令要興建一些道路，並須滿足以下條件：

- 每條道路連接兩個不同的城市。
- 任何 3 個城市均不能組成「孤立群」（即當中任何兩個城市均沒有道路連接）。
- 任何 20 個城市均不能組成「觀光圈」（即可以從 20 個城市的其中一個出發，經所建的道路到訪其餘 19 個城市，當中不會重複到訪城市也不會經過其他城市，最後返回起點）。

後來，該國的一名數學家證明了國王的要求是不可能達到的。求 n 的最小可能值。

There are n cities in a country, where $n > 20$. The king decided that some roads are to be built subject to the following requirements.

- Each road connects two different cities.
- Any 3 cities must not form an ‘isolated group’ (which means no two cities are connected by a road).
- Any 20 cities must not form a ‘sightseeing loop’ (which means one can start from one of the 20 cities and travel along the roads built to visit the other 19 cities without visiting the same city twice nor passing through other cities, and finally go back to the starting city).

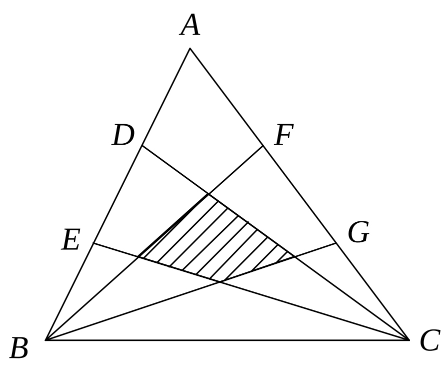
It was later proved by a mathematician in the country that the requirements of the king cannot be fulfilled. Find the smallest possible value of n.

19. 在一個遊戲中，11 人圍圈而坐，另外有 11 張卡紙，其中 8 張是空白的，另外 3 張則分別寫上「A」、「B」和「C」。開始時，每人隨機獲發一張卡紙。在每個回合中，每人都會把手上的卡紙傳給右方的人，然後主持人會要求知道「A」、「B」和「C」三張卡紙分別在誰人手上者舉手。結果在首三個回合中均沒有人舉手，而在第四回合中則有一人舉手。若第五和第六回合分別有 x 人和 y 人舉手，求乘積 xy。（各人都不會看到別人手上的卡紙，而且各人都是聰明的，即有足夠資料時便能作出推論。）

In a game, 11 people sat in a circle. There were 11 cards, 8 of which were blank, and the numbers ‘A’, ‘B’ and ‘C’ were written on the other 3 respectively. One card was distributed to each person at random at the beginning. In each round, each person passed the card to the person on his right, and then the players would be asked to raise his hands if he could tell which people were holding the cards with ‘A’, ‘B’ and ‘C’ written respectively. It turned out that nobody raised hands during the first three rounds, while one person raised hands during the fourth round. During the fifth and sixth rounds, there were x and y people raising hands respectively. Find the product xy. (The people could not see the cards held by others. They are also intelligent, so that deductions can be made whenever sufficient information is available.)

20. 圖中，ABC 是面積爲 1 的三角形。D、E 是 AB 上的點，F、G 則是 AC 上的點，使得 $AD = DE = EB$ 和 $AF = FG = GC$。求 BF、BG、CD 和 CE 圍成的區域的面積。

In the figure, ABC is a triangle with area 1. D, E are points on AB while F, G are points on AC such that $AD = DE = EB$ and $AF = FG = GC$. Find the area of the region bounded by BF, BG, CD and CE.



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