第十八屆培正數學邀請賽（2019 年）

18th Pui Ching Invitational Mathematics Competition (2019)

初賽（高中組）

Heat Event (Senior Secondary)

時限：1 小時 15 分

Time allowed: 1 hour 15 minutes

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 作答時，每題的答案均須以 0 至 9999 之間的整數表示。依照答題紙上的指示填寫答案，毋須呈交計算步驟。

Each answer must be given in the form of an integer between 0 and 9999. Follow the instructions on the answer sheet to enter the answers. You are not required to hand in your steps of working.

(d) 不得使用計算機。

The use of calculators is not allowed.

(e) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：每題的答案均須以 0 至 9999 之間的整數表示，如有需要應以上述範圍內最接近正確答案的整數回答。如有兩個這樣的整數與正確答案同樣接近，則以「四捨五入」的原則取較大的整數。請細閱答題紙上的指示。

Note: Each answer must be given in the form of an integer between 0 and 9999. Where necessary, the answer should be rounded off to the nearest integer in the above range. Read the instructions on the answer sheet in detail.

1. 有多少個不超過 10 的正整數 n 滿足 $-6n \le -42$？（3 分）

   How many positive integers n not exceeding 10 satisfy $-6n \le -42$? (3 marks)

2. 有多少個四位數可以寫成某正整數的四次方？（3 分）

   How many four-digit numbers can be written as the fourth power of a positive integer? (3 marks)

3. 一個公差為 2 的等差數列最多可以有多少個連續的質數項？（3 分）

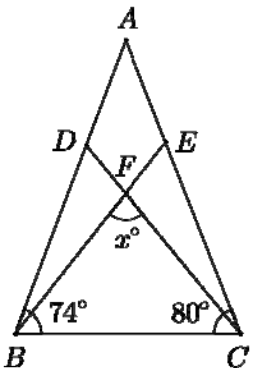
   What is the maximum number of consecutive prime number terms in an arithmetic sequence with common difference 2? (3 marks)

4. 設 a、b、c、d 為正整數。若 $2^{a^b} \cdot (2^c)^d$ 是立方數，求 $a+b+c+d$ 的最小可能值。（3 分）

   Let a, b, c and d be positive integers. If $2^{a^b} \cdot (2^c)^d$ is a cube, find the minimum possible value of $a+b+c+d$. (3 marks)

5. 在 ΔABC 中，D 和 E 分別是 AB 和 AC 上的點，使得 $DB = BC = CE$。BE 和 CD 相交於 F。若 $\angle ABC = 74°$、$\angle ACB = 80°$ 及 $\angle BFC = x°$，求 x 的值。（4 分）

   In ΔABC, D and E are points on AB and AC respectively such that $DB = BC = CE$. BE meets CD at F. If $\angle ABC = 74°$, $\angle ACB = 80°$ and $\angle BFC = x°$, find the value of x. (4 marks)



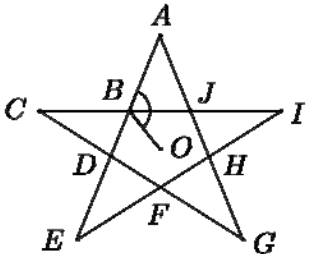
6. 某班 20 名學生的班號分別是 1 至 20，他們被分成 10 組，每組 2 人。若任意一組的兩名學生的班號之和均不超過 S，求 S 的最小可能值。（4 分）

There are 20 students in a class, with class numbers 1 to 20 respectively. They are divided into 10 groups of 2 students each. If the sum of class numbers of the two students in any group does not exceed S, find the smallest possible value of S. (4 marks)

7. 諾陶有 20 天假期，他計劃每連續 4 天工作後休息 1 天，並在工作天每天花 2 小時做作業，這樣他可以剛好及時完成假期作業。然而，諾陶一直在耽擱，沒有工作，直至他發現假期只餘下 4 天。假設諾陶以相同的速度去做作業。為了能及時完成所有作業，諾陶需要在餘下 4 天的每天平均花多少小時做作業？（4 分）

Omega has a 20-day holiday and plans to take a break of 1 day after every 4 consecutive days of work, and to spend 2 hours on assignments during each working day. Under this plan, Omega could just finish the assignments on time. However, he kept procrastinating and did not start working until he finds that there are only 4 days left in the holiday. Suppose Omega does the assignments at the same pace. How many hours on average must Omega spend on each of the remaining 4 days in order to finish all the assignments on time? (4 marks)

8. 現有一個星形 $ABCDEFGHIJ$，其中頂點 A、C、E、G、I 是一個正五邊形的頂點，而頂點 B、D、F、H、J 則是 $ACEGI$ 的對角線的交點。設 O 是該星形的中心。若 $\angle OBA = x°$，求 x 的值。（4 分）



There is a star shape $ABCDEFGHIJ$. The vertices A, C, E, G, I are the vertices of a regular pentagon, while the vertices B, D, F, H, J are the intersections of the diagonals of $ACEGI$. Let O be the centre of the star shape. If $\angle OBA = x°$, find the value of x. (4 marks)

9. 黑板上有 n 個正整數，其中首 9 個分別是 1 至 9，之後有若干個是 5，最後的若干個則是 8。若這些正整數的平均值為 $\frac{32}{5}$，求 n 的最小可能值。（5 分）

There are n positive integers on the blackboard. The first 9 of them are 1 to 9. Then there are some copies of 5 followed by some copies of 8 at the end. If the mean of all numbers on the blackboard is $\frac{32}{5}$, find the smallest possible value of n. (5 marks)

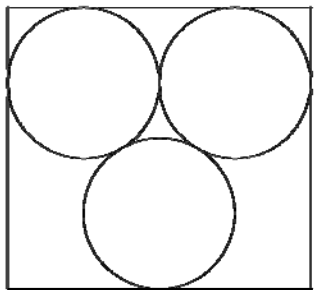
10. 現有 2019 顆勻稱的骰子，當中每顆骰子的六個面分別寫上 0 至 5 的整數。甲投擲其中一顆骰子並記下擲出的點數，之後乙投擲全部 2019 顆骰子並記下所得點數之和除以 6 的餘數。若二人所記下的數相同的概率以最簡分數表示時為 $\frac{a}{b}$，求 $a+b$ 的值。（5 分）

There are 2019 fair dice, each with the integers from 0 to 5 written on the six faces. Ann throws one of the dice and records the number obtained, while Ben throws all 2019 dice and records the remainder when the sum of the numbers obtained is divided by 6. If the probability that they record the same number is $\frac{a}{b}$ in lowest form, find the value of $a+b$. (5 marks)

11. 設 k 為實數。若方程 $x^2-kx+128=0$ 的兩根均為正整數，求 k 的所有可能值之和。（5 分）

Let k be a real number. If both roots of the equation $x^2-kx+128=0$ are positive integers, find the sum of all possible values of k. (5 marks)

12. 如圖所示，三個半徑相同的圓形位於一個長方形內。這些圓形兩兩互相外切，其中兩個圓形與長方形的兩條邊相切，另一個圓形則與長方形的一條邊相切。若長方形的面積為 $10000(2+\sqrt{3})$，求每個圓形的半徑。（5 分）



As shown in the figure, three circles with equal radii lie inside a rectangle. The circles are pairwise externally tangent to each other. Two of the circles are tangent to two sides of the rectangle, while the other circle is tangent to one side of the rectangle. If the area of the rectangle is $10000(2+\sqrt{3})$, find the radius of each circle. (5 marks)

13. 在首 2019 個正整數中，有多少個在二進制表示中包含恰好 7 個數字「1」？（6 分）

How many of the first 2019 positive integers contain exactly 7 copies of digit ‘1’ in binary notation? (6 marks)

14. 現有兩個袋子，當中各裝有編號為 1 至 9 的 9 個球。之後從每個袋子中隨機拿走一個球，並設 x 和 y 分別是兩個袋子中餘下的球的編號之和。若 $x-y$ 是 4 的倍數的概率以最簡分數表示時為 $\frac{a}{b}$，求 $a+b$ 的值。（6 分）

There are two bags, each with 9 balls labelled from 1 to 9. Now one ball is picked and removed from each bag randomly. Let x and y be the sums of labels of the remaining balls in the two bags respectively. If the probability that $x-y$ is a multiple of 4 is $\frac{a}{b}$ in lowest form, find the value of $a+b$. (6 marks)

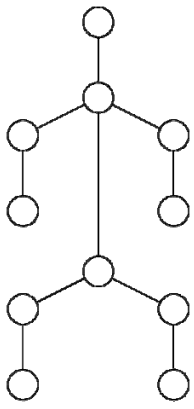
15. 設 a 為實數。若方程 $(x-6)(x-10)+2y^2+24y+a=0$ 有剛好一組實數解 (x,y)，求 a 的值。（6 分）

Let a be a real number. If the equation $(x-6)(x-10)+2y^2+24y+a=0$ has exactly one real solution (x,y), find the value of a. (6 marks)

16. 現有 12 張咭，其中紅、黃、藍、綠色各 3 張。現隨機抽出 3 張咭，若當中每出現一張紅色咭，我們就把它拿走，並隨機抽出另一張咭代替，直至所有抽出的咭均不是紅色為止。設 p 為最後得到的 3 張咭顏色相同的概率。求 $\frac{1}{p}$ 的值。（6 分）

There are 12 cards, with 3 of each of red, yellow, blue and green colours. We randomly pick 3 cards, with the rule that whenever a red card appears, it has to be removed and replaced by another card picked randomly, until all the cards chosen are not red. Let p be the probability that the 3 cards obtained in the end are of the same colour. Find the value of $\frac{1}{p}$. (6 marks)

17. 現要在圖中的圓形內填上 1 至 11 的整數（不可重複）。如果兩個圓形由直線連接著，則位處較高的圓形內的整數必須大於位處較低的圓形內的整數。那麼，有多少種方法可以完成要求？（7 分）



The integers from 1 to 11 are to be filled into the circles in the figure without repetition. It is required that if two circles are connected by a line, then the integer in the higher circle must be greater than that in the lower circle. How many ways are there to complete this task? (7 marks)

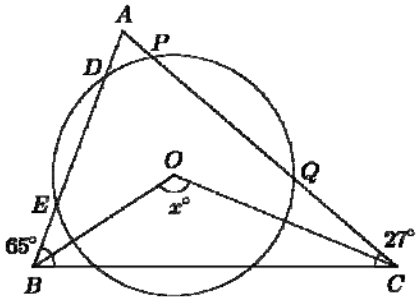
18. 設 P 是 ΔABC 所在的平面上的一點，使得 P 到直線 BC、CA 和 AB 的距離相同。若 $\angle BAC = 72°$ 及 $\angle BPC = x°$，求 x 的所有可能值之和。（7 分）

Let P be a point on the plane of ΔABC such that the distances from P to the lines BC, CA and AB are the same. If $\angle BAC = 72°$ and $\angle BPC = x°$, find the sum of all possible values of x. (7 marks)

19. 哥哥和弟弟在某星期的星期一至星期六都在嘗試解數學題。哥哥在星期一開始時立即解了一道題，弟弟則在星期一正午解了一道題。已知哥哥在任意連續 24 小時中都解了最少一道題，而弟弟則在任意連續 18 小時中解了最少一道題。此外，在星期一、星期三和星期五，首先解出第一道題的是哥哥；在星期二、星期四和星期六，首先解出第一道題的是弟弟。那麼，在這六天中，兄弟二人合共最少解了多少道題？（7 分）

A pair of brothers kept solving mathematical problems from Monday to Saturday in a certain week. The elder brother solved one problem immediately at the beginning of Monday, while the younger brother solved one at noon on Monday. It is known that the elder brother solved at least one problem during any continuous 24-hour period, while the younger brother solved at least one problem during any continuous 18-hour period. Furthermore, it was the elder brother who first solved a problem on Monday, Wednesday and Friday, while it was the younger brother who first solved a problem on Tuesday, Thursday and Saturday. What is the minimum total number of problems solved by the brothers during these six days? (7 marks)

20. 在 ΔABC 中，$\angle ABC = 65°$ 及 $\angle ACB = 27°$。一個以 O 為圓心的圓形與 AB 相交於 D 和 E，並與 AC 相交於 P 和 Q。若 $AD \times AE = BD \times BE$、$AP \times AQ = CP \times CQ$ 及 $\angle BOC = x°$，求 x 的值。（7 分）



In ΔABC, $\angle ABC = 65°$ and $\angle ACB = 27°$. A circle with centre O intersects AB at D and E and intersects AC at P and Q. If $AD \times AE = BD \times BE$, $AP \times AQ = CP \times CQ$ and $\angle BOC = x°$, find the value of x. (7 marks)

全卷完

END OF PAPER