第十八屆培正數學邀請賽（2019 年）

18th Pui Ching Invitational Mathematics Competition (2019)

決賽（高中組）

Final Event (Senior Secondary)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. A 和 B 是平面上兩個距離為 5 的點。有多少條直線與 A 的距離為 2，且與 B 的距離為 3？

   A and B are two points on the plane that are at a distance 5 apart. How many straight lines are at a distance of 2 from A and a distance of 3 from B?

2. 現有一個邊長為 4 和 6 的長方形，我們將這個長方形分成 n 個小長方形（它們的大小不一定相同），使得每個小長方形的邊長比例均為 $2:1$（或 $1:2$）。求 n 的最小可能值。

   There is a rectangle with side lengths 4 and 6. We divide this rectangle into n small rectangles (which does not necessarily have the same size) such that the side lengths of each small rectangle are in the ratio $2:1$ (or $1:2$). Find the smallest possible value of n.

3. 求最大的正整數 n，使得 n 的數位均不相同且不為零，另外每個數位均不是 n 的因數。

   Find the largest positive integer n such that the digits of n are all distinct and non-zero, and each digit is not a factor of n.

4. 設 x 為實數，使得 $3^{2x+2}\cdot 4^{x+1}-12\cdot 6^{x}+1=0$。求 x 的值。

   Let x be a real number such that $3^{2x+2}\cdot 4^{x+1}-12\cdot 6^{x}+1=0$. Find the value of x.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 求 $\dfrac{1}{50^{2019}}$ 以小數形式表示時的最後一個非零數位。

   Find the last non-zero digit in the decimal expansion of $\dfrac{1}{50^{2019}}$.

6. 現有 6 個大小不同的環和 3 支顏色不同的垂直棒。有多少種方法可以將每個環套在其中一支棒上，使得沒有環位於另一個較小的環上，且每支棒上均最少有 1 個環？

   There are 6 rings of different sizes and 3 vertical sticks of different colours. In how many ways can we put each ring on one of the sticks so that no ring is put on top of a smaller one, and each stick contains at least 1 ring?

7. 設 $f(x)=x^2+4x+2$ ，求方程 $f(f(x))=0$ 的所有實根之和。

   Let $f(x)=x^2+4x+2$. Find the sum of all real roots to the equation $f(f(x))=0$.

8. 現有 k 個不同的正整數，它們之和是 101。若它們當中任意兩個的最大公因數均等於 1，求 k 的最大可能值。

   There are k distinct positive integers with sum 101. If the H.C.F. of any two of them is 1, find the greatest possible value of k.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 給定乘式 $\text{ABAABA}=\text{BB}\times\text{CC}\times\text{CCDB}$（其中 ABAABA 是一個六位數，如此類推）。若每個字母代表一個由 1 至 9 的不同數字，求 ABCD 所代表的四位數。

   The expression $\text{ABAABA}=\text{BB}\times\text{CC}\times\text{CCDB}$ is given. (Here ABAABA represents a six-digit number, and similar for the others.) If each letter represents a different digit from 1 to 9, find the value of the four-digit number represented by ABCD.

10. 在一個遊戲中，玩家每局勝出的概率為 $\frac{2}{3}$。如果玩家在某局落敗會被扣 2 分，勝出則獲加分：如果玩家在上一局亦勝出則可加 3 分，否則加 2 分。求經過 10 局遊戲後玩家的分數的期望值。

In a game, the player has a probability of $\frac{2}{3}$ of winning in each round. If the player loses in a round, he would get 2 points deducted; if he wins in a round, he would be awarded 3 points if he also won in the previous round and 2 points otherwise. Find the expected score of the player after 10 rounds.

11. 某袋子中有 10 個球，編號為 1 至 10。若我們隨機抽出 7 個球，所得的球中有 5 個的編號是連續數的概率是多少？

In a bag there are 10 balls, numbered 1 to 10. If 7 balls are drawn at random, what is the probability that 5 balls with consecutive numbers are drawn?

12. 某三角形的邊長分別為 3、4 和 5。一個半徑為 r 的圓形的圓心位於三角形的邊界上，且這個圓與三角形的兩邊相切。求 r 的所有可能值之和。

A triangle has side lengths 3, 4 and 5. A circle of radius r has its centre lying on the boundary of the triangle and is tangent to two sides of the triangle. Find the sum of all possible values of r.

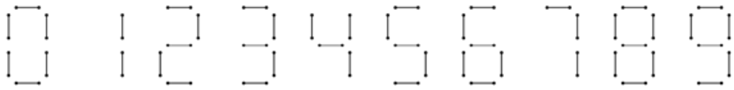
第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 求最小的正整數 n，使得 $\sqrt{n+2\sqrt{n+2\sqrt{n}}} > 28$。

Find the smallest positive integer n for which $\sqrt{n+2\sqrt{n+2\sqrt{n}}} > 28$.

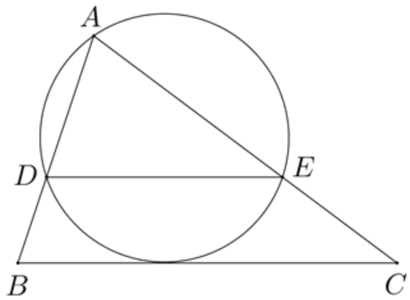
14. 如下圖所示，我們可以用火柴組成數字 0 至 9。例如：我們可用 12 支火柴組成整數 123。若 n 是不超過 10000 的正整數，且組成 n 所需的火柴數目等於組成 $n+2$ 所需的火柴數目，則 n 有多少個不同可能值？



As shown in the figure above, we can use some matches to form the digits 0 to 9. For example, we can use a total of 12 matches to form the integer 123. If n is a positive integer not exceeding 10000 such that the number of matches needed to form the number n is the same as the number of matches needed to form $n+2$, how many different possible values of n are there?

15. 設 D 和 E 分別為 ΔABC 的邊 AB 和 AC 上的點，使得 DE 和 BC 平行，且 ΔADE 的外接圓與 BC 相切。若 $AB=6$、$BC=10$ 及 $CA=9$，求 DE 的長度。

Let D and E be points on the sides AB and AC of ΔABC respectively such that DE and BC are parallel and the circumcircle of ΔADE is tangent to BC. If $AB=6$, $BC=10$ and $CA=9$, find the length of DE.



16. 設 A 為 3×3 矩陣，使得 $A\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}4\\5\\6\end{pmatrix}$ 及 $A^T\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}7\\8\\x\end{pmatrix}$。求 x 的值。

Let A be a 3×3 matrix such that $A\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $A^T\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}7\\8\\x\end{pmatrix}$. Find the value of x.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 求 $1^{24}+2^{24}+3^{24}+\cdots+24^{24}$ 除以 125 時的餘數。

Find the remainder when $1^{24}+2^{24}+3^{24}+\cdots+24^{24}$ is divided by 125.

18. 一部機器的顯示屏上有一條位於直角座標平面上的直線，其斜率為 m，其 x 截距亦是 m。設 S 為以 (–6,6)、(–6,0)、(6,0) 和 (6,6) 為頂點的長方形。當 m 的值連續地從 -100 增至 100 時，顯示屏上的直線隨之移動，並劃過 S 的一部分。求 S 被直線劃過的區域的面積。

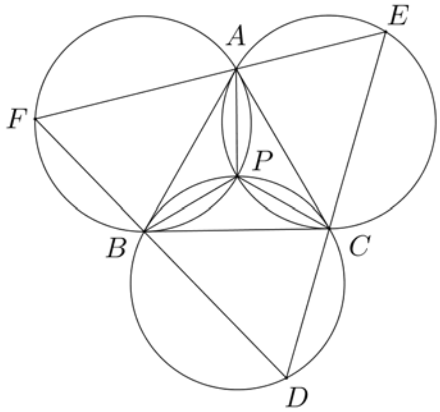
The screen of a machine displays a straight line on the Cartesian plane with slope m and x-intercept m. Let S be the rectangle with vertices (–6,6), (–6,0), (6,0) and (6,6). When the value of m increases from -100 to 100 continuously, the straight line on the screen moves and sweeps through a part of S. What is the area of the part of S swept by the line?

19. 5 名小童參加一個遊戲。在遊戲開始時，每人手上有一個球，然後每人寫下自己的一個「目標」，這個目標可以是任何一位遊戲參加者（包括自己）。在遊戲的每個回合中，每名小童同時把自己手上的所有球交給自己的目標。事後，每位小童都發現，自己在第 1 回合結束後擁有的球的數目與第 2 回合結束後擁有的球的數目相同。那麼，各人寫下的目標有多少個不同的可能組合？

5 children play a game. At the beginning, each child has a ball and individually writes down a ‘target’, which can be any one of the participants of the game (including himself). During each round of the game, each child passes all the balls in his hand to his target at the same time. After the game, every child finds that he has as many balls at the end of Round 1 as at the end of Round 2. How many different sets of possible combinations are there for the targets written down by the children?

20. 設 P 是 ΔABC 內一點，使得 $\angle APB = \angle BPC = 120°$ 及 $AP = 8$、$BP = 3$、$CP = 4$。分別在 ΔBPC、ΔCPA 和 ΔAPB 的外接圓上選取點 D、E 和 F，使得 DCE、EAF 和 FBD 均為直線。求 ΔDEF 的面積的最大可能值。

Let P be a point inside ΔABC such that $\angle APB = \angle BPC = 120°$ and $AP = 8$, $BP = 3$, $CP = 4$. Points D, E and F are chosen on the circumcircles of ΔBPC, ΔCPA and ΔAPB respectively such that DCE, EAF and FBD are straight lines. Find the greatest possible area of ΔDEF.



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