第十八屆培正數學邀請賽（2019 年）

18th Pui Ching Invitational Mathematics Competition (2019)

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. A 和 B 是平面上兩個距離為 5 的點。有多少條直線與 A 的距離為 2，且與 B 的距離為 3？

   A and B are two points on the plane that are at a distance 5 apart. How many straight lines are at a distance of 2 from A and a distance of 3 from B?

2. 一次足球賽共有 7 隊參加，當中任何兩隊均需要互相對賽一次，且每隊每天只能參加最多一場比賽。最少需要多少天才可完成整項比賽？

   There are 7 teams in a football tournament. Every two teams have to play against each other once, and each team may play at most one match on each day. At least how many days are required to finish the tournament?

3. 若 p 和 q 是不同的奇質數，則 p^2q+pq^2 最少有多少個不同的質因數？

   If p and q are distinct odd prime numbers, at least how many different prime factors does p^2q+pq^2 have?

4. 一個菱形的邊長為 20，其中一條對角線的長度為 24，求這個菱形的面積。

   The side length of a rhombus is 20 and the length of a diagonal is 24. Find the area of the rhombus.

第 5 至第 8 題，每題 4 分。

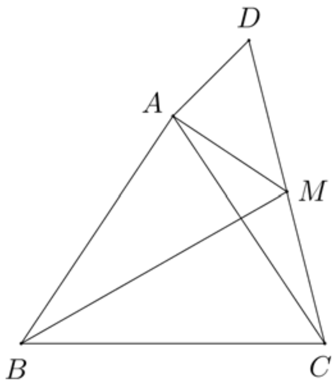
Questions 5 to 8 each carries 4 marks.

5. 設 a 和 b 為實數。若 $\frac{b}{ab+1}=\frac{1}{6}$ 及 $\frac{b^2}{a^2b^2+1}=\frac{1}{28}$，求 $\frac{a}{b}$ 的值。

   Let a and b be real numbers. If $\frac{b}{ab+1}=\frac{1}{6}$ and $\frac{b^2}{a^2b^2+1}=\frac{1}{28}$, find the value of $\frac{a}{b}$.

6. 在凸四邊形 $ABCD$ 中，M 是 CD 的中點。若 ΔADM、ΔABM 和 ΔBCM 的面積分別是 123、456 和 789，求 ΔABC 的面積。

   In convex quadrilateral $ABCD$, M is the mid-point of CD. If the areas of ΔADM, ΔABM and ΔBCM are 123, 456 and 789 respectively, find the area of ΔABC.



7. 凱因打算在未來六個月中學習組合學、三角學、幾何、複數、微分和積分 6 個數學單元。她將於每個月學習一個單元，而學習複數前必須先完成三角學和幾何兩個單元，且學習積分前必須先完成微分單元。那麼，凱因學習這 6 個單元的次序有多少種不同的可能性？

   Daphne plans to study 6 mathematics modules, namely, combinatorics, trigonometry, geometry, complex numbers, differentiation and integration, in the next six months. She will complete one module each month. She must finish trigonometry and geometry before studying complex numbers, and she must finish differentiation before studying integration. In how many different orders can Daphne finish the 6 modules?

8. 設 m 和 n 為整數，使得 $m=(4^{2n+1}\cdot 8^{n-2})^{\frac{1}{n}}$。求 m 的最大可能值。

   Let m and n be integers such that $m=(4^{2n+1}\cdot 8^{n-2})^{\frac{1}{n}}$. Find the greatest possible value of m.

第 9 至第 12 題，每題 5 分。

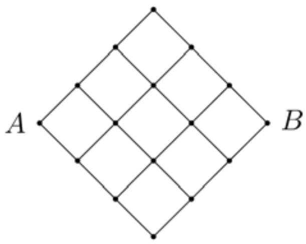
Questions 9 to 12 each carries 5 marks.

9. 有多少對不超過 100 的正整數 (m,n) 滿足 $\dfrac{m+n}{m-n}>2$？

   How many pairs (m,n) of positive integers not exceeding 100 satisfy $\dfrac{m+n}{m-n}>2$?

10. 如圖所示，一隻螞蟻從 A 點出發，每步均沿線段爬行至一個未曾到過的相鄰點。若牠在 8 步後到達 B 點，則牠的爬行路徑有多少個不同的可能性？

    As shown in the figure, an ant starts from point A and crawls along the line segments to an adjacent point which has not been travelled before in each step. If it reaches point B after 8 steps, how many different possibilities are there for the route it has taken?



11. 在一個遊戲中，如果玩家在某局落敗會被扣 2 分，勝出則獲加分：如果玩家在上一局亦勝出則可加 3 分，否則加 2 分。已知只要玩家每局勝出的概率相同且大於 p，則玩家的分數的期望值長遠而言必然是非負的。求 p 的最小可能值。

    In a game, if the player loses in a round, he would get 2 points deducted; if he wins in a round, he would be awarded 3 points if he also won in the previous round and 2 points otherwise. It is known that as long as the probability that the player wins in each round is the same and is greater than p, the expected score of the player is definitely non-negative in the long run. Find the smallest possible value of p.

12. 設 a 和 b 為實數。若函數 $f(x)=|x^2+ax+b|$ 在 $-1\le x\le 1$ 區間內的極大值為 c，求 c 的最小可能值。

    Let a and b be real numbers. If the function $f(x)=|x^2+ax+b|$ has maximum value c in the interval $-1\le x\le 1$, find the smallest possible value of c.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 設 $x_0 = 0$ 、 $x_1 = 1$ 及對 $n \ge 0$ 定義 $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ 。求 $\lim_{n\to\infty} x_n$ 的值。

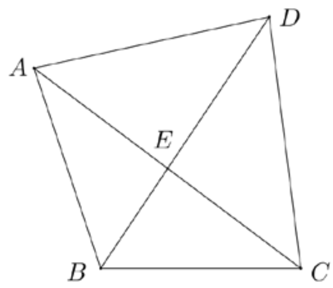
    Let $x_0 = 0$, $x_1 = 1$ and define $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for $n \ge 0$. Find $\lim_{n\to\infty} x_n$.

14. 設 *a* 和 *b* 為兩位數。若我們把 *b* 放在 *a* 的右方，可組成一個四位數 *n*。若 *n* 能被積 *ab* 整除，求 *a* 的所有可能值之和。

    Let *a* and *b* be two-digit numbers. If we put *b* on the right of *a*, a four-digit number *n* will be formed. If *n* is divisible by the product *ab*, find the sum of all possible values of *a*.

15. 在凸四邊形 *ABCD* 中， $AB = 12$ 、 $BC = 9$ 、 $CD = 13$ 、 $DA = 14$ 及 $AC = 15$。設 *E* 為對角線 *AC* 和 *BD* 的交點。求 $\frac{BE}{ED}$ 的值。

    In convex quadrilateral *ABCD*, $AB = 12$, $BC = 9$, $CD = 13$, $DA = 14$ and $AC = 15$. Let *E* be the intersection of the diagonals *AC* and *BD*. Find the value of $\frac{BE}{ED}$.



16. 黑板上寫有一個四位數 *n* 的所有正因數。若黑板上剛好一半的數能被 6 整除，則 *n* 有多少個不同的可能值？

    All the positive factors of a four-digit number *n* are written on the board. If exactly half of the numbers on the board are divisible by 6, how many possible values of *n* are there?

第 17 至第 20 題，每題 7 分。

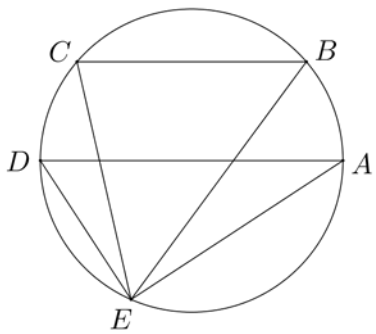
Questions 17 to 20 each carries 7 marks.

17. 設 $x_1 = 2019$ 及對 $n \ge 1$ 定義 $x_{n+1} = \sqrt{x_n + 8} - \sqrt{x_n + 3}$ 。求不超過 $x_1 + x_2 + \cdots + x_{2019}$ 的最大整數。

    Let $x_1 = 2019$ and define $x_{n+1} = \sqrt{x_n + 8} - \sqrt{x_n + 3}$ for $n \ge 1$. Find the greatest integer not exceeding $x_1 + x_2 + \cdots + x_{2019}$.

18. 設 A、B、C、D 和 E 依次為某圓上的五點，且 AD 為該圓的直徑，並與 BC 平行。若 $AE=8$ 及 $DE=BC=6$，求 $EB\times EC$ 的值。

Let A, B, C, D and E be five points on a circle in this order. Furthermore, suppose AD is a diameter of the circle that is parallel to BC. If $AE=8$ and $DE=BC=6$, find the value of $EB\times EC$.



19. 我們在 20190504 的連續數位中加入一個或多個「+」號，可得到不同的算式，例如 $2019+0504$ 和 $20+190+5+04$（我們將「0504」視作「504」，如此類推）。所得的算式有多少個不同的可能值？

By inserting one or more '+' signs between the digits of 20190504, one gets different expressions such as $2019+0504$ and $20+190+5+04$ (we regard '0504' as the same as '504' and so on). How many different possible values are there for such an expression?

20. 設 n 為正整數。我們從小至大列出所有介乎 0 和 1 之間，且分母小於 n 的最簡分數。其中兩個相鄰的分數是 $\frac{a}{b}$ 和 $\frac{99}{100}$，當中 a 和 b 都是四位數，而 a 的百位和十位分別是 5 和 4。求 b 的值。

Let n be a positive integer. We list out all fractions between 0 and 1 in lowest form with denominator less than n in ascending order. Two adjacent fractions are $\frac{a}{b}$ and $\frac{99}{100}$, where a and b are four-digit numbers, and the hundreds digit and tens digit of a are 5 and 4 respectively. Find the value of b.

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END OF PAPER