第十八屆培正數學邀請賽（2019 年）

18th Pui Ching Invitational Mathematics Competition (2019)

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

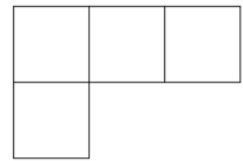The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 如圖所示，一個 L 形方塊由四個單位正方形組成。若我們將單位正方形替換成邊長為 3 的正方形以放大 L 形方塊，則 L 形方塊的周長增加了多少單位？

   An L-tetromino is formed by four unit squares as shown. If we enlarge the L-tetromino by using squares of side length 3 instead of unit squares, by how many units is the perimeter of the L-tetromino increased?

2. 求最大的整數 n，使得 $\sqrt[n]{4^{33} \cdot 6^{66} \cdot 9^{55}}$ 是整數。

   Find the largest integer n for which $\sqrt[n]{4^{33} \cdot 6^{66} \cdot 9^{55}}$ is an integer.

3. 求最大的正整數 n，使得 n 的數位均不相同且不為零，另外每個數位均不是 n 的因數。

   Find the largest positive integer n such that the digits of n are all distinct and non-zero, and each digit is not a factor of n.

4. 一個菱形的邊長為 20，其中一條對角線的長度為 24，求這個菱形的面積。

   The side length of a rhombus is 20 and the length of a diagonal is 24. Find the area of the rhombus.

5. 一個九邊形（不一定是凸多邊形）的九隻內角組成一個等差數列。若其中兩隻內角是 164° 和 $x°$，求 x 的最小可能值。

    The nine interior angles of a nonagon (not necessarily convex) form an arithmetic sequence. If two of the interior angles are 164° and $x°$, find the smallest possible value of x.

6. 有多少對不超過 100 的正整數 (m,n) 滿足 $0 < \dfrac{m+n}{m-n} < 2$？

    How many pairs (m,n) of positive integers not exceeding 100 satisfy $0 < \dfrac{m+n}{m-n} < 2$？

7. 如下圖所示，我們可以用火柴組成數字 0 至 9。如果正整數 n 可寫成 $a+b$（其中 a 和 b 都是正整數），且組成 n 所需的火柴數目等於組成 a 和 b 所需的火柴總數，則我們稱 n 為「好數」。例如：由於 10 可寫成 $9+1$，而組成 10、9 和 1 分別需要 8、6 和 2 支火柴，因此 10 是「好數」。若 n 是大於 10 的「好數」，求 n 的最小可能值。
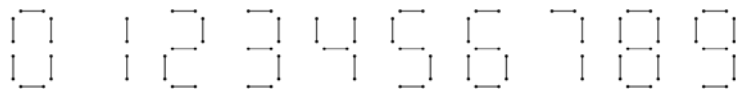
    As shown in the figure above, we can use some matches to form the digits 0 to 9. A positive integer n is said to be 'good' if it can be written as $a+b$ (where a and b are positive integers) such that the number of matches needed to form the number n is the same as the total number of matches needed to form a and b. For example, the number 10 is 'good' since it can be written as $9+1$, whereas we need 8, 6 and 2 matches to form the numbers 10, 9 and 1 respectively. If n is a 'good' number greater than 10, find the smallest possible value of n.

8. 有多少種不同的三角形的每隻內角（以「度」為單位時）均是整數？（若兩個三角形相似，則視作同一種三角形。）

    How many different kinds of triangles have all interior angles being integers (in degrees)? (Two triangles are regarded to be of the same kind if they are similar to each other.)

9. 設 *ABCD* 是凸四邊形，使得 $\angle DAB = \angle DBC = 90°$ 及 $DA = AB = 6$。設 *E* 是對角線 *AC* 和 *BD* 的交點，若 $BE = 2\sqrt{2}$ ，求 *ABCD* 的面積。

    Let *ABCD* be a convex quadrilateral such that $\angle DAB = \angle DBC = 90°$ and $DA = AB = 6$. Let *E* be the intersection of the diagonals *AC* and *BD*. If $BE = 2\sqrt{2}$, find the area of *ABCD*.
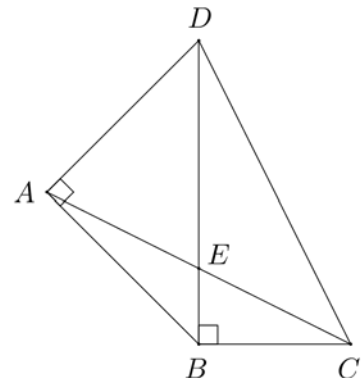
10. 現有 *k* 個正整數（可以相同），它們之和是 101。若它們當中任意兩個的最大公因數均大於 1，求 *k* 的最大可能值。

    There are *k* positive integers (not necessarily distinct) with sum 101. If the H.C.F. of any two of them is greater than 1, find the greatest possible value of *k*.

11. 紙張上有一個等腰直角三角形。我們從直角向對邊畫出高，可把三角形分成兩個全等的小三角形。對小三角形重複以上步驟，經過 4 次後我們可將原來的三角形分成 16 個全等的小三角形。那麼，有多少種方法從所得的圖中選出兩個不同的三角形（可能有重疊部分），且它們為相似三角形？

    There is a right-angled isosceles triangle on a piece of paper. By drawing the altitude from the right angle to the opposite side, the triangle would be divided into two identical smaller triangles. We can repeat this process on the smaller triangles. By doing this 4 times in total, the original triangle would have been divided into 16 identical smaller triangles. How many ways are there to select two different triangles (possibly with overlapping regions) in the resulting figure that are similar to each other?
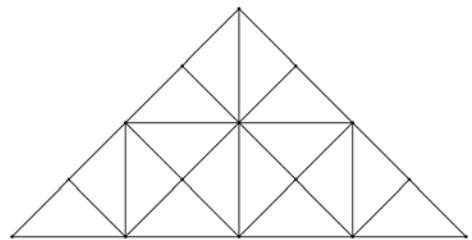
12. 現有 8 個球，我們隨機將每個球放進 3 個箱其中之一。求至少一個箱裝有恰好 3 個球的概率。

There are 8 balls and each of them is randomly put into one of the 3 boxes. Find the probability that there is at least one box containing exactly 3 balls.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 求最小的正整數 n，使得 $\sqrt{n+2\sqrt{n+2\sqrt{n}}} > 28$。

Find the smallest positive integer n for which $\sqrt{n+2\sqrt{n+2\sqrt{n}}} > 28$.

14. 設 n 為小於 10 的正整數。現有 n 張分別編號為 1 至 n 的咭片，若把它們隨機排列成一個 n 位數，該 n 位數可被 n 整除的概率為 $\dfrac{1}{n}$。求 n 的所有可能值之和。

Let n be a positive integer less than 10. There are n cards numbered 1 to n. If they are randomly permuted to form an n-digit number, the probability that the number formed is divisible by n is $\dfrac{1}{n}$. Find the sum of all possible values of n.

15. 現有一個 2×3 方塊，它的 6 個單位方格的顏色各不相同。有多少種不同的方法將這個方塊分成一塊或多塊？（我們只能沿着單位方格的公共邊進行分割。）

There is a 2×3 tile such that the colours of the 6 unit cells are distinct. How many different ways are there to divide this tile into one or more pieces? (We can only divide along the common edges of the unit cells.)

16. 有多少組非零整數 (a,b,c) 滿足 $\dfrac{8}{abc}+\dfrac{1}{a}+\dfrac{2}{b}+\dfrac{1}{c}=1+\dfrac{2}{bc}+\dfrac{1}{ac}+\dfrac{2}{ab}$ ？

How many triples (a,b,c) of non-zero integers satisfy $\dfrac{8}{abc}+\dfrac{1}{a}+\dfrac{2}{b}+\dfrac{1}{c}=1+\dfrac{2}{bc}+\dfrac{1}{ac}+\dfrac{2}{ab}$?

17. 求 $\dfrac{1}{75}+\dfrac{17}{75\times76}+\dfrac{17\times18}{75\times76\times77}+\dfrac{17\times18\times19}{75\times76\times77\times78}+\cdots$ 的值。

Find the value of $\dfrac{1}{75}+\dfrac{17}{75\times76}+\dfrac{17\times18}{75\times76\times77}+\dfrac{17\times18\times19}{75\times76\times77\times78}+\cdots$.

18. 在 ΔABC 中，$AB=4$，$BC=3$ 及 $CA=5$。在 ΔXYZ 中，$XY=XZ$，$YZ=3$，且從 X 到 YZ 的高為 4。若 B、C、Y、Z 位於同一直線上（不一定按此順序），求 ΔABC 和 ΔXYZ 的重疊部分的面積的最大可能值。

In ΔABC, $AB=4$, $BC=3$ and $CA=5$. In ΔXYZ, $XY=XZ$, $YZ=3$ and the height from X to YZ is 4. If B, C, Y, Z lie on the same straight line (not necessarily in this order), find the greatest possible area of the overlapping region of ΔABC and ΔXYZ.

19. 我們在 20190504 的連續數位中加入一個或多個「+」號，可得到不同的算式，例如 $2019+0504$ 和 $20+190+5+04$（我們將「0504」視作「504」，如此類推）。所得的算式有多少個不同的可能值？

By inserting one or more '+' signs between the digits of 20190504, one gets different expressions such as $2019+0504$ and $20+190+5+04$ (we regard '0504' as the same as '504' and so on). How many different possible values are there for such an expression?

20. 設 n 為正整數。我們從小至大列出所有介乎 0 和 1 之間，且分母小於 n 的最簡分數。其中兩個相鄰的分數是 $\dfrac{a}{b}$ 和 $\dfrac{99}{100}$，當中 a 和 b 都是四位數，而 a 的百位和十位分別是 5 和 4。求 b 的值。

Let n be a positive integer. We list out all fractions between 0 and 1 in lowest form with denominator less than n in ascending order. Two adjacent fractions are $\dfrac{a}{b}$ and $\dfrac{99}{100}$, where a and b are four-digit numbers, and the hundreds digit and tens digit of a are 5 and 4 respectively. Find the value of b.