第十八屆培正數學邀請賽（2019 年）

18th Pui Ching Invitational Mathematics Competition (2019)

決賽（中二組）

Final Event (Secondary 2)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 現有一個邊長為 4 和 6 的長方形，我們將這個長方形分成 n 個小長方形（它們的大小不一定相同），使得每個小長方形的邊長比例均為 $2:1$（或 $1:2$）。求 n 的最小可能值。

   There is a rectangle with side lengths 4 and 6. We divide this rectangle into n small rectangles (which does not necessarily have the same size) such that the side lengths of each small rectangle are in the ratio $2:1$ (or $1:2$). Find the smallest possible value of n.

2. 對任意正實數 x 和 y，定義 $x\#y=\frac{1}{x}+\frac{1}{y}$。若把整數 1、2、3、4 以某種次序填入算式 (☐#☐)#(☐#☐) 的方格中，求所得結果的最小可能值。

   For any positive real numbers x and y, define $x\#y=\frac{1}{x}+\frac{1}{y}$. We put the integers 1, 2, 3, 4 into the cells of the expression (☐#☐)#(☐#☐) in some order. Find the smallest possible value of the result obtained.

3. 現有 6 個大小不同的環和 3 支顏色不同的垂直棒。有多少種方法可以將每個環套在其中一支棒上，使得沒有環位於另一個較小的環上？

   There are 6 rings of different sizes and 3 vertical sticks of different colours. In how many ways can we put each ring on one of the sticks so that no ring is put on top of a smaller one?

4. 若 p 和 q 是不同的奇質數，則 p^2q+pq^2 最少有多少個不同的質因數？

   If p and q are distinct odd prime numbers, at least how many different prime factors does p^2q+pq^2 have?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 現有 k 個正整數，它們之和是 101。若 $k>1$ 且它們當中任意兩個的最大公因數均大於 1，求 k 的最小可能值。

   There are k positive integers with sum 101. If $k>1$ and the H.C.F. of any two of them is greater than 1, find the smallest possible value of k.

6. 求 $\frac{1}{50^{2019}}$ 以小數形式表示時的最後一個非零數位。

   Find the last non-zero digit in the decimal expansion of $\frac{1}{50^{2019}}$.

7. 一個凸九邊形的九隻內角組成一個等差數列。若其中兩隻內角是 $164°$ 和 $x°$，求 x 的最小可能值。

   The nine interior angles of a convex nonagon form an arithmetic sequence. If two of the interior angles are $164°$ and $x°$, find the smallest possible value of x.

8. 某國家設有 8% 銷售稅，所有貨品的標價均包含稅款在內。貨品的稅前價格（以元為單位）必須是整數，而包含稅款在內的標價為四捨五入後的最接近整數。例如，若某物件的稅前價格是 926 元，那麼包含稅款在內的標價就會是 1000 元（將 926 乘以 1.08 後取至最接近整數而得）。若某正整數可以是一個包含稅款在內的標價，那麼我們稱它為「好數」，故由上述例子知 1000 是「好數」。那麼，在首 2019 個正整數當中，有多少個是「好數」？

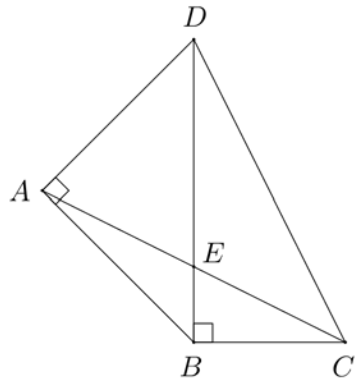
   In a country, there is an 8% sales tax. All goods on sale will bear a price tag with the tax-inclusive price. The price net of tax (in dollars) must be an integer, and the tax-inclusive price will be rounded off to the nearest integer. For example, for an article with a net price of 926 dollars, the tax-inclusive price will be 1000 dollars (which is obtained by multiplying 926 by 1.08 and then rounding off to the nearest integer). A positive integer is said to be ‘good’ if it may appear as a tax-inclusive price, so 1000 is ‘good’ in view of the previous example. How many of the first 2019 positive integers are ‘good’?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 設 $ABCD$ 是凸四邊形，使得 $\angle DAB = \angle DBC = 90°$ 及 $DA = AB = 6$。設 E 是對角線 AC 和 BD 的交點，若 $BE = 2\sqrt{2}$，求 $ABCD$ 的面積。

   Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle DBC = 90°$ and $DA = AB = 6$. Let E be the intersection of the diagonals AC and BD. If $BE = 2\sqrt{2}$, find the area of $ABCD$.



10. 6 名小童參加一個遊戲。在遊戲開始時，每人手上有一個球，然後每人寫下自己的一個「目標」，這個目標可以是任何一位遊戲參加者（包括自己）。在遊戲的每個回合中，每名小童同時把自己手上的所有球交給自己的目標。在第 2 回合結束後，每名小童均得到遊戲開始時自己的那個球。那麼，各人寫下的目標有多少個不同的可能組合？

    6 children play a game. At the beginning, each child has a ball and individually writes down a 'target', which can be any one of the participants of the game (including himself). During each round of the game, each child passes all the balls in his hand to his target at the same time. At the end of Round 2, every child gets his ball at the beginning of the game. How many different sets of possible combinations are there for the targets written down by the children?

11. 考慮直角座標平面上以 (0,0)、(4,0)、(4,4) 和 (0,4) 為頂點的正方形。設 k 為實數，而正方形內位於兩直線 $x - 3y = k$ 和 $x - 3y = k + 2$ 之間的區域記為 S。求 S 的面積的最大可能值。

    Consider the square with vertices (0,0), (4,0), (4,4) and (0,4) on the Cartesian plane. Let k be a real number and denote by S the region inside the square and between the straight lines $x - 3y = k$ and $x - 3y = k + 2$. Find the greatest possible area of S.

12. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 ABCD 所代表的四位數的所有可能值之和。

    In the multiplication shown, each letter represents a different digit from 0 to 9. Find the sum of all possible values of the four-digit number represented by ABCD.

$$\begin{array}{rcccc} & & & \mathrm{A} & \mathrm{B} \\ \times & & \mathrm{A} & \mathrm{C} & \mathrm{A} \\ \hline & \mathrm{D} & \mathrm{A} & \mathrm{D} & \mathrm{A} \end{array}$$

第 13 至第 16 題，每題 6 分。

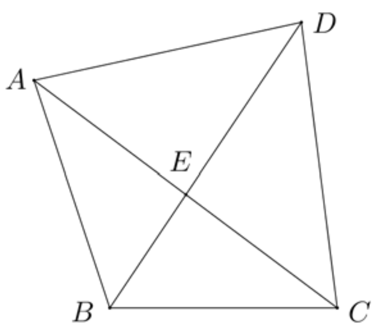
Questions 13 to 16 each carries 6 marks.

13. 現有一個 $2\times n$ 的表格，每格中均寫有一個不同的正整數。若把每個直行的兩個數加起來，所得的和分別是 2019, 2020, …, $2018+n$。求 n 的最大可能值。

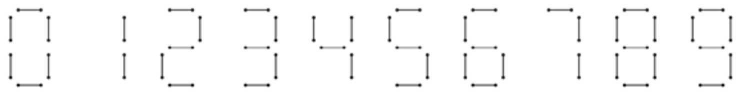
There is a $2\times n$ table. In each cell a different positive integer is written. If we add up the two numbers in each column, the sums obtained are 2019, 2020, …, $2018+n$ respectively. Find the greatest possible value of n.

14. 在凸四邊形 $ABCD$ 中，$AB=15$、$BC=20$、$CD=CA=25$ 及 $DA=14$。設 E 為對角線 AC 和 BD 的交點，求 $\frac{BE}{ED}$ 的值。

In convex quadrilateral $ABCD$, $AB=15$, $BC=20$, $CD=CA=25$ and $DA=14$. Let E be the intersection of the diagonals AC and BD. Find the value of $\frac{BE}{ED}$.



15. 如下圖所示，我們可以用火柴組成數字 0 至 9。例如：我們可用 12 支火柴組成整數 123。若 n 是不超過 10000 的正整數，且組成 n 所需的火柴數目等於組成 $n+2$ 所需的火柴數目，則 n 有多少個不同可能值？



As shown in the figure above, we can use some matches to form the digits 0 to 9. For example, we can use a total of 12 matches to form the integer 123. If n is a positive integer not exceeding 10000 such that the number of matches needed to form the number n is the same as the number of matches needed to form $n+2$, how many different possible values of n are there?

16. 黑板上寫有一個四位數 n 的所有正因數。若黑板上剛好一半的數能被 6 整除，則 n 有多少個不同的可能值？

All the positive factors of a four-digit number n are written on the board. If exactly half of the numbers on the board are divisible by 6, how many possible values of n are there?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 x、y 和 z 為互不相同的實數。若

$$(x+y-7)[z(x+y)+24]=(y+z-7)[x(y+z)+24]=(z+x-7)[y(z+x)+24]，$$

求 $x^2+y^2+z^2$ 的值。

Let x, y and z be pairwise distinct real numbers. If

$$(x+y-7)[z(x+y)+24]=(y+z-7)[x(y+z)+24]=(z+x-7)[y(z+x)+24],$$

find the value of $x^2+y^2+z^2$.

18. 設 $a_0=a_{53}=0$。有多少種方法可以把 1、2、⋯、52 排列成 $(a_1,a_2,\ldots,a_{52})$，使得 a_1-a_0、a_2-a_1、⋯、$a_{53}-a_{52}$ 這 53 個差中，最多只有兩個不同的值？

Let $a_0=a_{53}=0$. How many ways are there to permute 1, 2, …, 52 into $(a_1,a_2,\ldots,a_{52})$ so that there are at most two different values among the 53 differences a_1-a_0, a_2-a_1, …, $a_{53}-a_{52}$?

19. 現有一個 3×3 方塊，它的 9 個單位方格的顏色各不相同。有多少種不同的方法將這個方塊分成一塊或多塊？（我們只能沿着單位方格的公共邊進行分割。）

There is a 3×3 tile such that the colours of the 9 unit cells are distinct. How many different ways are there to divide this tile into one or more pieces? (We can only divide along the common edges of the unit cells.)

20. 設 $x_1=2019$ 及對 $n\geq1$ 定義 $x_{n+1}=\sqrt{x_n+8}-\sqrt{x_n+3}$。求不超過 $x_1+x_2+\cdots+x_{2019}$ 的最大整數。

Let $x_1=2019$ and define $x_{n+1}=\sqrt{x_n+8}-\sqrt{x_n+3}$ for $n\geq1$. Find the greatest integer not exceeding $x_1+x_2+\cdots+x_{2019}$.

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END OF PAPER