第十七屆培正數學邀請賽（**2018** 年）

17th Pui Ching Invitational Mathematics Competition (2018)

決賽（高中組）

Final Event (Senior Secondary)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 一隻螞蟻從一個正方體的其中一個頂點出發，沿着正方體的棱爬行。假設牠未曾在同一條棱上爬行兩次，求這隻螞蟻所經過的棱的數目的最大可能值。

   An ant started at a vertex of a cube and crawled along the edges of the cube. Suppose it did not crawl along the same edge twice. Find the greatest possible number of edges that the ant passed through.

2. 一部計算機上有一個特殊的紅色按鈕。若在顯示屏上的數值是某正整數 n 時按下紅色按鈕，顯示的數值將會改變成 n 的各個數位的平方之和。例如，若顯示的數值是 2018，按下紅色按鈕後顯示屏上的數值會改變成 $2^2+0^2+1^2+8^2=69$。若顯示屏上最初的數值是 1001，那麼按下紅色按鈕 2018 次後會得到甚麼數值？

   There is a special red button on a calculator. If the red button is pressed when the number on the screen is a positive integer n, the number displayed will become the sum of the squares of the digits of n. For example, if 2018 is displayed and the red button is pressed, the number on the screen will become $2^2+0^2+1^2+8^2=69$. If the number on the screen is 1001 initially, what number will be obtained after pressing the red button 2018 times?

3. 冠霖家中有一棵特別的植物，它起初的高度為 70。從第一天起，每當冠霖在第 k 天為植物澆水時，植物的高度會增加 k。但到了每天晚上，植物的高度會減少 5。假設冠霖每天都為植物澆水一次，且他在第 n 天早上澆水後發現植物的高度超過 200。求 n 的最小可能值。

   Zachary has a special plant at home. Its initial height is 70. Starting from the first day, when Zachary waters the plant on the kth day, the height of the plant will increase by k. But when it comes to the night of each day, the height of the plant will decrease by 5. Suppose Zachary waters the plant once every day. He finds that the height of the plant exceeds 200 after watering the plant on the morning of the n-th day. Find the smallest possible value of n.

4. 若多項式 $P(x)$ 除以 $(x-2)^2$ 時的餘數是 $2x-7$，求 $P(x)$ 除以 $x-2$ 時的餘數。

If the remainder when a polynomial $P(x)$ is divided by $(x-2)^2$ is $2x-7$, find the remainder when $P(x)$ is divided by $x-2$.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 已知 a, b, c, d, e 組成一個等差數列，其中 $a<b<c<d<e$，且 a, b, e 組成一個等比數列。若 $c=125$，求 e 的值。

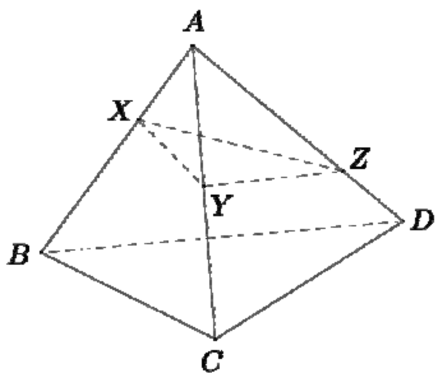
The numbers a, b, c, d, e form an arithmetic sequence. Furthermore, $a<b<c<d<e$ and a, b, e form a geometric sequence. If $c=125$, find the value of e.

6. 現將整數 1、2、3、4、5 和 6 寫在一個六個面的顏色兩兩不同的正方體的面上，使得每個面有一個不同的整數，且每組對面上的整數之和是 7。那麼，共有多少種不同的寫法？

The integers 1, 2, 3, 4, 5 and 6 are written on the faces of a cube whose six faces have pairwise distinct colours so that each face contains a distinct integer, and the sum of integers on each pair of opposite faces is 7. How many different possible ways are there to write the integers?

7. 設 $ABCD$ 為一個體積為 1 的四面體，設 X、Y 和 Z 分別為邊 AB、AC 和 AD 上的點，使得 $AX:XB=1:2$、$AY:YC=1:1$ 及 $AZ:ZD=2:1$。求四面體 $AXYZ$ 的體積。

Let $ABCD$ be a tetrahedron with volume 1. Let X, Y and Z be points on sides AB, AC and AD respectively such that $AX:XB=1:2$, $AY:YC=1:1$ and $AZ:ZD=2:1$. Find the volume of the tetrahedron $AXYZ$.



8. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 ABCD 所代表的四位數。

In the addition shown, each letter represents a different digit from 0 to 9. Find the four-digit number represented by ABCD.

$$\begin{array}{rccccc} & & A & B & B & C \\ + & & B & C & A & B \\ \hline & D & D & A & D & D \end{array}$$

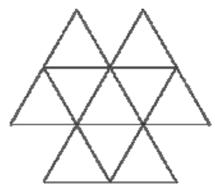
第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 某校有 2018 名學生，他們的學生編號分別是 1 至 2018。他們被分成若干班，使得對任意質數 *p* 和正整數 *a*，編號為 *a* 和 *pa* 的學生都不會同班。那麼，該校最少有多少班？

There are 2018 students in a school, numbered 1 to 2018. They are divided into several classes in a way such that for any prime number p and positive integer a, the students with numbers a and pa will not belong to the same class. What is the minimum number of classes in the school?

10. 考慮如圖所示由 10 個等邊三角形組成的形狀。現將其中一些等邊三角形塗上綠色（可不把任何三角形塗色，或把全部三角形塗色），然後在每個等邊三角形中寫下與這個三角形相鄰的綠色三角形的數目（若兩個三角形其中一邊重合，則視兩者為相鄰；一個三角形並不與自己相鄰），再計算形狀中寫下的所有數之和。那麼，有多少種塗色的方法，可使得這個和是一個奇數？（不允許將形狀旋轉。）



Consider the shape formed by 10 equilateral triangles as shown. Now some (possibly none or all) of the equilateral triangles are coloured green. Next, in each of the equilateral triangles, one writes down the number of green triangles adjacent to this triangle. (Two triangles are adjacent if they share a common side. A triangle is not adjacent to itself.) The sum of all numbers written down in the shape is then computed. How many ways of colouring are there such that the sum is an odd number? (Rotation of the shape is not allowed.)

11. 設 $n=48\times 49\times\cdots\times 147\times\left(\frac{1}{48}+\frac{1}{49}+\cdots+\frac{1}{147}\right)$，且 k 是正整數，使得 10^k 是 n 的因數。求 k 的最大可能值。

Let $n=48\times 49\times\cdots\times 147\times\left(\frac{1}{48}+\frac{1}{49}+\cdots+\frac{1}{147}\right)$, and k be a positive integer such that 10^k is a factor of n. Find the largest possible value of k.

12. 現有一個八位數 n，由兩個「1」、兩個「2」、一個「3」和三個「4」組成。已知「3」出現在兩個「1」之間（不一定連續），而且兩個「2」之間共有偶數個數字（可以是 0 個）。那麼，n 有多少個不同的可能值？

There is an eight-digit number n, which is formed by two ‘1’s, two ‘2’s, one ‘3’ and three ‘4’s. It is known that the digit ‘3’ appears between the two ‘1’s (not necessarily consecutive), and there is an even number (possibly 0) of digits between the two ‘2’s. How many different possible values of n are there?

第 13 至第 16 題，每題 6 分。

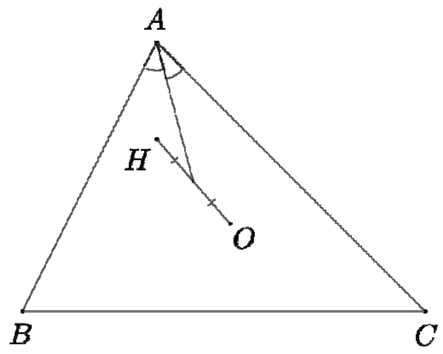
Questions 13 to 16 each carries 6 marks.

13. 一個邊長為 1 的正方體以如下方式被放置，其中只有一個頂點 V 接觸着水平的桌面，而 V 對面的頂點恰好位於 V 的正上方。現有一個水平平面將正方體分成兩個體積相同的部分，求正方體被平面所截的橫切面面積。

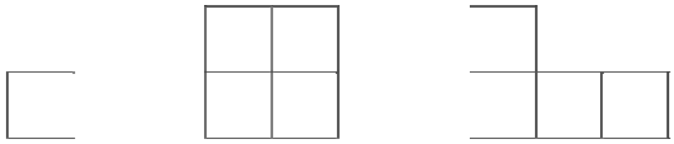
A cube of side length 1 is aligned in such a way that only one of the vertices V touches a horizontal table and the opposite vertex of V lies exactly above V. A horizontal plane cuts the cube into two halves of equal volume. Find the area of the cross-section of the cube cut by the plane.

14. 設 H 和 O 分別為 ΔABC 的垂心和外心，假設 $\angle BAC$ 的內角平分線平分 OH。若 $AB=30$ 及 $AC=48$，求 BC 的長度。

Let H and O be the orthocentre and the circumcentre of ΔABC respectively. Suppose the internal angle bisector of $\angle BAC$ bisects OH. If $AB=30$ and $AC=48$, find the length of BC.



15. 我們須用 7 塊咭片完整地覆蓋一個 5×5 的方格表。這 7 塊咭片中，其中一塊是一個 1×1 的正方形，其餘六塊則是 2×2 的正方形或由 4 個方格組成的 L 形（見下圖；使用 L 形時可把它們反射和旋轉）。對於 5×5 方格表的一個方格，若存在一種覆蓋的方法使得 1×1 正方形被放在該個方格上，則稱它為「好格」。那麼共有多少個「好格」？



A 5×5 table is to be tiled using 7 cardboards, one of which is a 1×1 square while each of the other six is either a 2×2 square or an L-shape formed by 4 cells (see figure above; reflection and rotation of the L-shape are allowed). A cell of the 5×5 table is said to be 'good' if there exists a tiling such that the 1×1 square is put on that cell. How many 'good' cells are there?

16. 平面上有 12 點。畫出當中每一對點的垂直平分線後，我們發現沒有兩條垂直平分線互相平行，且這些垂直平分線共組成 n 個交點。求 n 的最大可能值。

There are 12 points on a plane. After drawing the perpendicular bisector of each pair of these points, it was found that no two of the perpendicular bisectors are parallel and they form a total of n points of intersections. Find the greatest possible value of n.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 對於正整數 n，設 $\omega(n)$ 代表 n 的不同質因數的數目。例如，由於 $40=2^3\times5$ 的質因數只有 2 和 5，我們有 $\omega(40)=2$。已知 $\omega(1)+\omega(2)+\cdots+\omega(1000)=2126$ 及 $\omega(1001)+\omega(1002)+\cdots+\omega(2000)=2328$，求

$$\omega(1^2)+\omega(1^2+2^2)+\omega(1^2+2^2+3^2)+\cdots+\omega(1^2+2^2+\cdots+1000^2)$$

的值。

Let $\omega(n)$ denote the number of distinct prime factors of a positive integer n. For example, since the only prime factors of $40=2^3\times5$ are 2 and 5, we have $\omega(40)=2$. Given that $\omega(1)+\omega(2)+\cdots+\omega(1000)=2126$ and $\omega(1001)+\omega(1002)+\cdots+\omega(2000)=2328$, find the value of

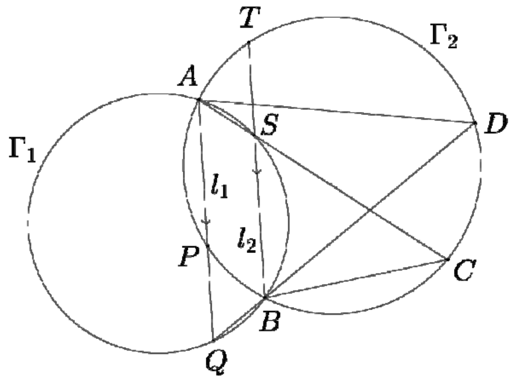
$$\omega(1^2)+\omega(1^2+2^2)+\omega(1^2+2^2+3^2)+\cdots+\omega(1^2+2^2+\cdots+1000^2).$$

18. 求以下方程組中 x_1 的值。

Find the value of x_1 in the following system of equations.

$$\begin{cases} C_0^0 x_0 + C_0^1 x_1 + C_0^2 x_2 + \cdots + C_0^{100} x_{100} = 1 \\ C_1^1 x_1 + C_1^2 x_2 + \cdots + C_1^{100} x_{100} = 1 \\ C_2^2 x_2 + \cdots + C_2^{100} x_{100} = 1 \\ \vdots \\ C_{100}^{100} x_{100} = 1 \end{cases}$$

19. 兩個圓形 Γ_1 和 Γ_2 相交於點 A 和 B，平行線 l_1 和 l_2 分別穿過 A 和 B。設 l_1 分別再交 Γ_1 和 Γ_2 於 Q 和 P，設 l_2 分別再交 Γ_1 和 Γ_2 於 S 和 T。AS 的延長線與 Γ_2 相交於 C，QB 的延長線與 Γ_2 相交於 D。若 $AP : PQ = 5 : 4$ 、 $BS : ST = 3 : 2$ 及 $BC = 12$ ，求 AD 的長度。



Two circles Γ_1 and Γ_2 intersect at points A and B. Parallel lines l_1 and l_2 pass through A and B respectively. Let l_1 meet Γ_1 and Γ_2 again at Q and P respectively. Let l_2 meet Γ_1 and Γ_2 again at S and T respectively. The extension of AS meets Γ_2 at C. The extension of QB meets Γ_2 at D. If $AP : PQ = 5 : 4$, $BS : ST = 3 : 2$ and $BC = 12$, find the length of AD.

20. 設 f 和 g 為無窮可微函數。對於曲線 $y = f(x)$ 上任意兩點 $P = (a, f(a))$ 和 $Q = (b, f(b))$ ，其中 $a < b$ ，曲線的弧 PQ 必定位於線段 PQ 之下，而且由弧 PQ 和線段 PQ 所圍成的區域的面積等於 $g(b-a)$ 。若 $g(1) = 10$ ，求 $g(5)$ 的值。

Let f and g be infinitely differentiable functions. For any two points $P = (a, f(a))$ and $Q = (b, f(b))$ on the curve $y = f(x)$ with $a < b$, the arc PQ of the curve always lies below the segment PQ, and the area of the region bounded by the arc PQ and the segment PQ is equal to $g(b-a)$. If $g(1) = 10$, find the value of $g(5)$.

全卷完

END OF PAPER