第十七屆培正數學邀請賽（**2018** 年）

17th Pui Ching Invitational Mathematics Competition (2018)

決賽（中四組）

Final Event (Secondary 4)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 若多項式 $P(x)$ 除以 $(x-2)^2$ 時的餘數是 36，求 $P(x)$ 除以 $x-2$ 時的餘數。

   If the remainder when a polynomial $P(x)$ is divided by $(x-2)^2$ is 36, find the remainder when $P(x)$ is divided by $x-2$.

2. 袋子裏裝有 *n* 顆糖果，其中 9 顆是橙色的，其餘是黃色的。卓穎從袋中隨機抽出 2 顆糖果。若 2 顆被取出的糖果均為橙色的概率是 $\frac{3}{10}$，求 *n* 的值。

   There are *n* sweets in a bag, where 9 of them are orange and the rest are yellow. Christie takes 2 sweets from the bag at random. If the probability that the 2 sweets taken are both orange is $\frac{3}{10}$, find the value of *n*.

3. 設 Γ_1 和 Γ_2 分別為半徑為 6 和 8 的兩個圓形。若 Γ_1 和 Γ_2 相交於點 *A* 和 *B*，求 *AB* 的長度的最大可能值。

   Let Γ_1 and Γ_2 be two circles with radii 6 and 8 respectively. If Γ_1 and Γ_2 intersect at points *A* and *B*, find the greatest possible value of the length of *AB*.

4. 黑板上寫着首 2018 個正整數，之後所有 3 的倍數被擦去，然後再寫上所有不超過 2018 的 6 的正倍數，最後所有 27 的倍數被擦去。這時黑板上有多少個整數？

   The first 2018 positive integers are written on the board. All multiples of 3 are then erased. Next, all positive multiples of 6 not exceeding 2018 are written. Finally, all multiples of 27 are erased. How many integers are there on the board in the end?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 設 n 為合成數，且 n 的每個正因數（除 n 本身外）均為一位數。求 n 的最大可能值。

   Let n be a composite number such that every positive factor of n (other than n itself) is a one-digit number. Find the greatest possible value of n.

6. 現將整數 1、2、3、4、5 和 6 寫在一個正方體的面上，使得每個面有一個不同的整數，且每組對面上的整數之和是 7。設 n 為其中擁有公共頂點的三個面上所寫的整數之和，求 n 的所有可能值之和。

   The integers 1, 2, 3, 4, 5 and 6 are written on the faces of a cube so that each face contains a distinct integer, and the sum of integers on each pair of opposite faces is 7. Let n be the sum of integers written on three faces which share a common vertex. Find the sum of all possible values of n.

7. 一個 16×16 的方格表被 n 個正方形覆蓋。每個正方形的大小都是 1×1 、5×5 、9×9 或 13×13 ，它們均完整地覆蓋方格，且完全位於方格表內，使得方格表的每個方格均被恰好一個正方形覆蓋着。求 n 的最小可能值。

   There is a 16×16 table which is covered by n squares. The size of each square is 1×1, 5×5, 9×9 or 13×13. Every square covers the cells completely and lies entirely inside the table, and each cell of the table is covered by exactly one square. Find the smallest possible value of n.

8. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 ABCD 所代表的四位數。

   In the addition shown, each letter represents a different digit from 0 to 9. Find the four-digit number represented by ABCD.

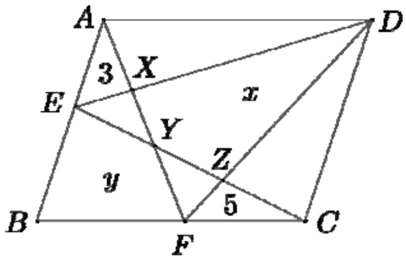
$$\begin{array}{rccccc} & & \text{A} & \text{B} & \text{B} & \text{C} \\ + & & \text{B} & \text{C} & \text{A} & \text{B} \\ \hline & \text{D} & \text{D} & \text{A} & \text{D} & \text{D} \end{array}$$

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 設 $ABCD$ 為平行四邊形，設 E 和 F 分別為邊 AB 和 BC 上的點。以 X 表示 AF 和 DE 的交點，Y 表示 AF 和 CE 的交點，Z 表示 DF 和 CE 的交點。若 AEX、CFZ、$DXYZ$ 和 $BFYE$ 的面積分別是 3、5、x 和 y，求 $x-y$ 的值。

   Let $ABCD$ be a parallelogram. Let E and F be points on sides AB and BC respectively. Denote by X the intersection of AF and DE, Y the intersection of AF and CE, and Z the intersection of DF and CE. If the areas of AEX, CFZ, $DXYZ$ and $BFYE$ are 3, 5, x and y respectively, find the value of $x-y$.



10. 設 $f(x)=4x^3-3x$。求 $f\left(f\left(\cdots f\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\cdots\right)\right)$ 的值，當中共有 2018 個 f。

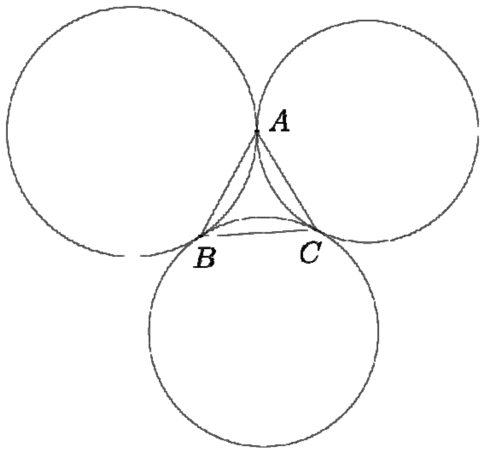
    Let $f(x)=4x^3-3x$. Find the value of $f\left(f\left(\cdots f\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)\cdots\right)\right)$ where there are 2018 copies of f's in the expression.

11. 若座標平面上的一條直線滿足以下條件，則我們稱它是「穩定的」：若 (a,b) 是直線上的一個點，則 (a^2,b^2) 亦位於這條直線上。那麼，共有多少條「穩定的」直線？

    A straight line in the coordinate plane is said to be 'stable' if it satisfies the following property: if (a,b) is a point on the straight line, then (a^2,b^2) also lies on this straight line. How many 'stable' straight lines are there?

12. 三個半徑分別為 1、2 和 3 的圓形兩兩互相外切，切點分別為 A、B 和 C。求 ΔABC 的面積。

    Three circles with radii 1, 2 and 3 respectively are pairwise externally tangent to each other with tangential points A, B and C respectively. Find the area of ΔABC.

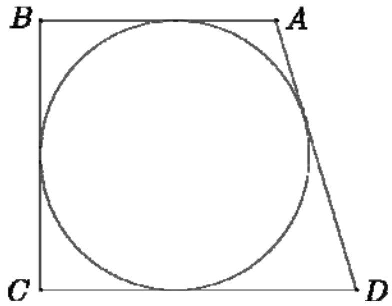


第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 設 $ABCD$ 為梯形，滿足 $\angle ABC = \angle BCD = 90°$ 、 $BC = 20$ 及 $CD = 2AB$ 。假設存在一個圓形內切於 $ABCD$，求 $ABCD$ 的面積。

    Let $ABCD$ be a trapezoid with $\angle ABC = \angle BCD = 90°$, $BC = 20$ and $CD = 2AB$. Suppose there is a circle inscribed in *ABCD*. Find the area of *ABCD*.



14. 有 2018 名小童圍成一圈坐下，開始時部分人在唱歌（可能是全部人或沒有人）。在每分鐘的開始時，對於每一名小童，若與他相鄰的兩人在上一分鐘均在唱歌，那麼這名小童就會唱歌，否則他就不會唱歌。在 10 天後的某分鐘，有 n 名小童在唱歌。求 n 的所有可能值之和。

    There are 2018 kids sitting in a circle. Initially, some of them (possibly all or none) are singing. At the beginning of every minute, a kid will sing if both of his neighbours were singing in the previous minute. Otherwise, the kid will not sing. After 10 days, n kids are singing in a certain minute. Find the sum of all possible values of n.

15. 設 k 為實數，使得方程 $x^2y^2 + 2x^2y + 3xy + 4x^2 + 5x + k = 0$ 有唯一實數解。求 k 的值。

    Let k be a real number such that the equation $x^2y^2 + 2x^2y + 3xy + 4x^2 + 5x + k = 0$ has a unique real solution. Find the value of k.

16. 平面上有 12 點。畫出當中每一對點的垂直平分線後，我們發現沒有兩條垂直平分線互相平行，且這些垂直平分線共組成 n 個交點。求 n 的最大可能值。

    There are 12 points on a plane. After drawing the perpendicular bisector of each pair of these points, it was found that no two of the perpendicular bisectors are parallel and they form a total of n points of intersections. Find the greatest possible value of n.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 對於正整數 n，設 $\omega(n)$ 代表 n 的不同質因數的數目。例如，由於 $40=2^3\times 5$ 的質因數只有 2 和 5，我們有 $\omega(40)=2$。已知 $\omega(1)+\omega(2)+\cdots+\omega(1000)=2126$ 及 $\omega(1001)+\omega(1002)+\cdots+\omega(2000)=2328$，求

$$\omega(1^2)+\omega(1^2+2^2)+\omega(1^2+2^2+3^2)+\cdots+\omega(1^2+2^2+\cdots+1000^2)$$

的值。

Let $\omega(n)$ denote the number of distinct prime factors of a positive integer n. For example, since the only prime factors of $40=2^3\times 5$ are 2 and 5, we have $\omega(40)=2$. Given that $\omega(1)+\omega(2)+\cdots+\omega(1000)=2126$ and $\omega(1001)+\omega(1002)+\cdots+\omega(2000)=2328$, find the value of

$$\omega(1^2)+\omega(1^2+2^2)+\omega(1^2+2^2+3^2)+\cdots+\omega(1^2+2^2+\cdots+1000^2).$$

18. 設 n 為不超過 50 的正整數。已知若函數 f 滿足以下兩個條件：

(1) 對任意正整數 k，$f(k)$ 皆是正整數，再者，

(2) 方程 $f(f(\cdots f(x)\cdots))=x$（其中 f 在方程的左方出現了 n 次）剛好有 50 個正整數解，

方程 $f(x)=x$ 則必定有最少一個正整數解。那麼，n 共有多少個可能值？

Let n be a positive integer not exceeding 50. It is known that, if a function f satisfies the following two conditions:

(1) $f(k)$ is a positive integer for every positive integer k, and

(2) the equation $f(f(\cdots f(x)\cdots))=x$, in which f occurs n times on the left hand side, has exactly 50 positive integer solutions,

then the equation $f(x)=x$ must have at least one positive integer solution. How many possible values of n are there?

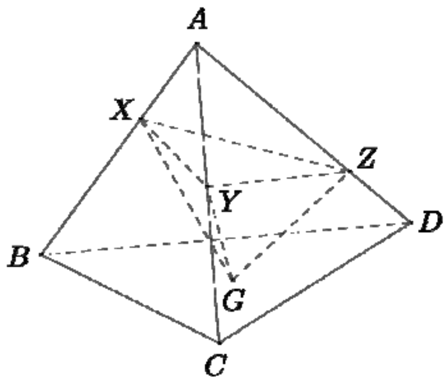
19. 求以下方程組中 x_1 的值。

Find the value of x_1 in the following system of equations.

$$\begin{cases} C_0^0 x_0 + C_0^1 x_1 + C_0^2 x_2 + \cdots + C_0^{100} x_{100} = 1 \\ C_1^1 x_1 + C_1^2 x_2 + \cdots + C_1^{100} x_{100} = 1 \\ C_2^2 x_2 + \cdots + C_2^{100} x_{100} = 1 \\ \vdots \\ C_{100}^{100} x_{100} = 1 \end{cases}$$

20. 設 $ABCD$ 是一個體積為 1 的四面體，設 X、Y 和 Z 分別是邊 AB 、AC 和 AD 上的點，使得 $AX:XB=1:2$ 、$AY:YC=1:1$ 及 $AZ:ZD=2:1$ 。設 G 是 ΔBCD 的重心，求四面體 $GXYZ$ 的體積。

Let $ABCD$ be a tetrahedron with volume 1. Let X, Y and Z be points on sides AB, AC and AD respectively such that $AX:XB=1:2$, $AY:YC=1:1$ and $AZ:ZD=2:1$. Let G be the centroid of ΔBCD. Find the volume of the tetrahedron $GXYZ$.



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