第十七屆培正數學邀請賽（2018 年）

17th Pui Ching Invitational Mathematics Competition (2018)

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 一隻螞蟻從一個正方體的其中一個頂點出發，沿着正方體的棱爬行。假設牠未曾在同一條棱上爬行兩次，求這隻螞蟻所經過的棱的數目的最大可能值。

   An ant started at a vertex of a cube and crawled along the edges of the cube. Suppose it did not crawl along the same edge twice. Find the greatest possible number of edges that the ant passed through.

2. 冠霖家中有一棵特別的植物，它起初的高度為 70。從第一天起，每當冠霖為植物澆水時，植物的高度會增加 12。但到了每天晚上，植物的高度會減少 5。假設冠霖每天都為植物澆水一次，且他在第 n 天早上澆水後發現植物的高度超過 200。求 n 的最小可能值。

   Zachary has a special plant at home. Its initial height is 70. Starting from the first day, when Zachary waters the plant, the height of the plant will increase by 12. But when it comes to the night of each day, the height of the plant will decrease by 5. Suppose Zachary waters the plant once every day. He finds that the height of the plant exceeds 200 after watering the plant on the morning of the n-th day. Find the smallest possible value of n.

3. 袋子裏裝有 n 顆糖果，其中 9 顆是橙色的，其餘是黃色的。卓穎從袋中隨機抽出 2 顆糖果。若 2 顆被取出的糖果均為橙色的概率是 $\frac{3}{10}$，求 n 的值。

   There are n sweets in a bag, where 9 of them are orange and the rest are yellow. Christie takes 2 sweets from the bag at random. If the probability that the 2 sweets taken are both orange is $\frac{3}{10}$, find the value of n.

4. 設 n 為四位數，且已知 n 是介乎 2300 和 2400 之間的一個 9 的倍數。若我們知道 n 的個位數字，仍無法推斷出 n 的值。若我們進一步知道 n 不是 4 的倍數，求 n 的值。

Let n be a four-digit number which is known to be a multiple of 9 between 2300 and 2400. If we are told the unit digit of n, we won't be able to deduce the value of n. If it is known further that n is not a multiple of 4, find the value of n.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 有多少個正整數數列共有 6 項，其首項和尾項分別是 1 和 10，且每項均比之前一項大？

How many sequences of positive integers are there such that there are 6 terms in total, that the first term and last term are 1 and 10 respectively, and that each term is greater than its previous term?

6. 已知 a, b, c, d, e 組成一個等差數列，其中 $a < b < c < d < e$，且 a, b, e 組成一個等比數列。若 $c = 125$，求 e 的值。

The numbers a, b, c, d, e form an arithmetic sequence. Furthermore, $a < b < c < d < e$ and a, b, e form a geometric sequence. If $c = 125$, find the value of e.

7. 設 $n = p \times q \times r$，其中 p 和 q 是不同的一位質數，r 是兩位質數，n 則是三位數。已知在 n、p、q 和 r 的數位之中，合共只有兩個不同的數字。求 n 的值。

Let $n = p \times q \times r$, where p and q are distinct one-digit prime numbers, r is a two-digit prime number while n is a three-digit number. Given that there are only two different numbers among the digits of n, p, q and r, find the value of n.

8. 已知從無窮數列 $a_1, a_2, a_3, \ldots$ 中，對任何正整數 n 皆有 $a_{n+1} = \dfrac{1}{1-a_n}$。若 k 是一個兩位數，且 a_1 和 a_k 均等於 k，求 k 的所有可能值之和。

It is known that in the infinite sequence $a_1, a_2, a_3, \ldots$, we have $a_{n+1} = \dfrac{1}{1-a_n}$ for any positive integer n. If k is a two-digit number, and both a_1 and a_k are equal to k, find the sum of all possible values of k.

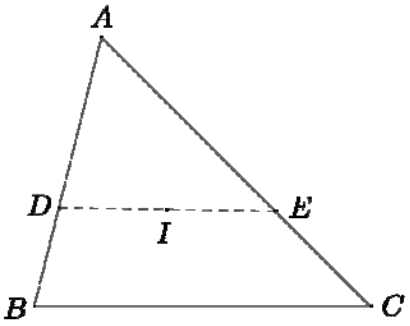
第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 若 n 是 146411024 的一個因數，當中 $3000 < n < 4000$ ，求 n 的值。

   If n is a factor of 146411024 where $3000 < n < 4000$, find the value of n.

10. 設 I 是 ΔABC 的內心，設 D 和 E 分別是邊 AB 和 AC 上的點，使得 DE 與 BC 互相平行，而且 DE 穿過 I。若 $AB = 6$ 、 $AC = 9$ 及 $BC = 10$ ，求 $\dfrac{[ADE]}{[ABC]}$ 的值，其中 $[XYZ]$ 代表 ΔXYZ 的面積。

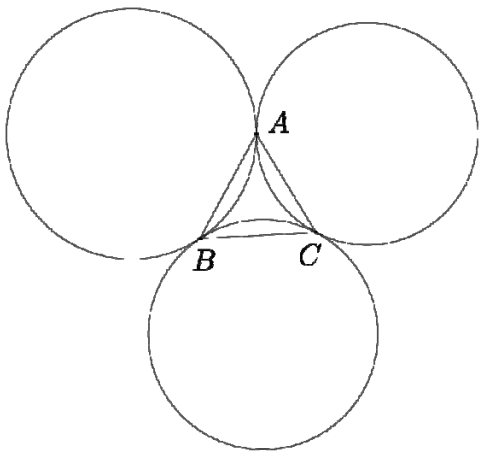


   Let I be the incentre of ΔABC. Let D and E be points on sides AB and AC respectively such that DE is parallel to BC and DE passes through I. If $AB = 6$, $AC = 9$ and $BC = 10$, find the value of $\dfrac{[ADE]}{[ABC]}$, where $[XYZ]$ denotes the area of ΔXYZ.

11. 現有一個八位數 n，由兩個「1」、兩個「2」、一個「3」和三個「4」組成。已知「3」出現在兩個「1」之間（不一定連續），而且兩個「2」之間共有偶數個數字（可以是 0 個）。那麼，n 有多少個不同的可能值？

    There is an eight-digit number n, which is formed by two ‘1’s, two ‘2’s, one ‘3’ and three ‘4’s. It is known that the digit ‘3’ appears between the two ‘1’s (not necessarily consecutive), and there is an even number (possibly 0) of digits between the two ‘2’s. How many different possible values of n are there?

12. 三個半徑分別為 1、2 和 3 的圓形兩兩互相外切，切點分別為 A、B 和 C。求 ΔABC 的面積。



    Three circles with radii 1, 2 and 3 respectively are pairwise externally tangent to each other with tangential points A, B and C respectively. Find the area of ΔABC.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 某校有 2018 名學生，他們的學生編號分別是 1 至 2018。他們被分成若干班，使得對任意質數 p 和正整數 a、b，編號為 a 和 $p^b a$ 的學生都不會同班。那麼，該校最少有多少班？

    There are 2018 students in a school, numbered 1 to 2018. They are divided into several classes in a way such that for any prime number p and positive integers a, b, the students with numbers a and $p^b a$ will not belong to the same class. What is the minimum number of classes in the school?

14. 現有 100 盞燈，編號分別是 1 到 100，每盞燈均連接着一個按鈕。當按鈕被按下時，對應的燈會由開轉成關或由關轉成開。開始時，所有燈都是關的。第一個小童按下每個按鈕一次，第二個小童按下每個對應編號為 2 的倍數的燈的按鈕兩次，第三個小童按下每個對應編號為 3 的倍數的燈的按鈕三次，如此類推，直至第 100 個小童按下對應編號為 100 的燈的按鈕 100 次後，共有多少盞燈是開的？

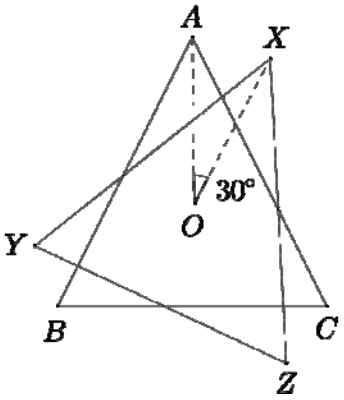
    There are 100 lamps labelled from 1 to 100, each of which is connected to a switch. When the switch is pressed, the corresponding lamp will change from on to off or from off to on. Initially, all the lamps are off. The first kid presses every switch once. The second kid presses every switch connected to a lamp with a label which is a multiple of 2 for two times. The third kid presses every switch connected to a lamp with a label which is a multiple of 3 for three times, and so on. After the 100th kid presses the switch connected to the lamp with label 100 for 100 times, how many lamps are on?

15. 設 n 和 k 為滿足 $C_k^n = 8008$ 的正整數。求 n 的所有可能值之和。

    Let n and k be positive integers satisfying $C_k^n = 8008$. Find the sum of all possible values of n.

16. 設 ΔABC 為邊長為 10 的等邊三角形，並以 ΔABC 的中心為旋轉中心將三角形順時針旋轉 30° 得到 ΔXYZ。求 ΔABC 和 ΔXYZ 重疊部分的面積。

    Let ΔABC be an equilateral triangle with side length 10. Using the centre of ΔABC as rotational centre, rotate the triangle clockwise by 30° to obtain ΔXYZ. Find the area of the overlapping region of ΔABC and ΔXYZ.



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在課室裏，老師對五名學生小陳、小李、小張、小王和小何說：「我寫下了一個數位兩兩不同的五位數 n。我會讓小陳看 n 的萬位和千位，讓小李看 n 的千位和百位，讓小張看 n 的百位和十位，讓小王看 n 的十位和個位，並讓小何看 n 的個位和萬位。」之後老師如所述般讓每名學生知道 n 的兩個數字，然後各人圍圈而坐，展開了以下的對話。

老師：「知道 n 的一個質因數的同學請舉手。」小王和小何舉手。

老師：「知道 n 的一個質因數的同學請舉手。」小陳、小王和小何舉手。

老師：「知道 n 的一個合成數因數的同學請舉手。」其中兩個同學舉手。

其中一個同學說：「那麼我知道 n 的值了，它是 7 的倍數。」

假設學生們都是聰明的，能按照其他學生的舉動作出正確分析。求 n 的值。

In a classroom, the teacher said to five students, Ann, Ben, Cat, Dan and Eva, 'I have written down a five-digit number n whose digits are pairwise distinct. I will let Ann see the ten thousands and thousands digits of n, let Ben see the thousands and hundreds digits of n, let Cat see the hundreds and tens digits of n, let Dan see the tens and unit digits of n and let Eva see the unit and ten thousands digits of n.' The teacher then let each student know two digits of n as said, and then everybody sat in a circle and started the following conversation.

'For those who know a prime factor of n, raise your hand,' the teacher said. Dan and Eva raised their hands.

'For those who know a prime factor of n, raise your hand,' the teacher said. Ann, Dan and Eva raised their hands.

'For those who know a composite factor of n, raise your hand,' the teacher said. Two of the students raised their hands.

'Then I know the value of n. It is a multiple of 7,' one of the students said.

Suppose the students are clever enough to analyse correctly what the other students have done. Find the value of n.

18. 設 $n \geq 3$ 為整數。現在平面上有 n 個點，畫出當中每一對點的垂直平分線後，我們發現沒有兩條垂直平分線互相平行，且這些垂直平分線共組成 k 個交點。設 a_n 表示 k 的最大可能值。在數列 $a_3, a_4, a_5, \ldots$ 中，最接近 2018 的一項的值是多少？

Let $n \geq 3$ be an integer. Now there are n points on a plane. After drawing the perpendicular bisector of each pair of these points, it was found that no two of the perpendicular bisectors are parallel and they form a total of k points of intersections. Let a_n denote the maximum value of k. What is the value of the term in the sequence $a_3, a_4, a_5, \ldots$ that is closest to 2018?

19. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[2.1] = 2$ 、$[4] = 4$ 和 $[5.7] = 5$ 。若 n 是正整數且 $\left[\frac{n}{\sqrt{2017}}\right] \neq \left[\frac{n}{\sqrt{2018}}\right]$，求 n 的最小可能值。

Let $[x]$ denote the largest integer not exceeding x. For example, $[2.1] = 2$, $[4] = 4$ and $[5.7] = 5$. If n is a positive integer and $\left[\frac{n}{\sqrt{2017}}\right] \neq \left[\frac{n}{\sqrt{2018}}\right]$, find the smallest possible value of n.

20. 設 A、B、C 和 D 為座標平面上四點，它們的 x-座標兩兩不同。若直線 AB、AC、AD、BC、BD 和 CD 的 y-截距分別是 2、3、5、7、11 和 k，求 k 的值。

Let A, B, C and D be four points in the coordinate plane with pairwise distinct x-coordinates. If the y-intercepts of lines AB, AC, AD, BC, BD and CD are 2, 3, 5, 7, 11 and k respectively, find the value of k.

全卷完

END OF PAPER