第十七屆培正數學邀請賽（2018 年）

17th Pui Ching Invitational Mathematics Competition (2018)

決賽（中二組）

Final Event (Secondary 2)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 現將整數 1、2、3、4、5 和 6 寫在一個正方體的面上，使得每個面有一個不同的整數，且每組對面上的整數之和皆等於 S。求 S 的值。

   The integers 1, 2, 3, 4, 5 and 6 are written on the faces of a cube so that each face contains a distinct integer, and the sum of integers on each pair of opposite faces is equal to S. Find the value of S.

2. 設 n 為正整數，且 $a=\frac{n}{5}$。小強在計算 $13a$ 的值時，先將 a 化成小數，卻誤把其小數部分略去才乘上 13。求小強所得的結果與正確答案之差的最大可能值。

   Let n be a positive integer and $a=\frac{n}{5}$. When Kenny tried to evaluate $13a$, he first converted a to a decimal number, but then he mistakenly omitted the decimal part before multiplying by 13. Find the greatest possible value of the difference between the result obtained by Kenny and the correct answer.

3. 設 Γ_1 和 Γ_2 分別為半徑為 6 和 8 的兩個圓形。若 Γ_1 和 Γ_2 相交於點 A 和 B，求 AB 的長度的最大可能值。

   Let Γ_1 and Γ_2 be two circles with radii 6 and 8 respectively. If Γ_1 and Γ_2 intersect at points A and B, find the greatest possible value of the length of AB.

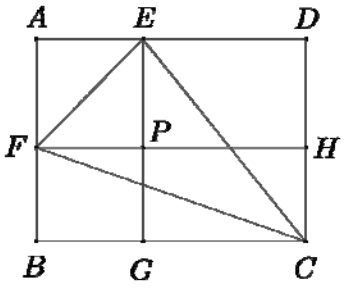
4. 某等差數列的首項為 5，每項皆是正整數，且其中兩項為 11 和 17。若該等差數列的第 100 項是 k，求 k 的所有可能值之和。

   In an arithmetic sequence, the first term is 5, each term is a positive integer and two of the terms are 11 and 17. If the 100th term of the arithmetic sequence is k, find the sum of all possible values of k.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 設 *ABCD* 為長方形，設 *E*、*F*、*G* 和 *H* 分別為邊 *DA*、*AB*、*BC* 和 *CD* 上的點，使得線段 *EG* 和 *FH* 將長方形分成四個較小的長方形。設 *P* 為 *EG* 和 *FH* 的交點。若 *ABCD* 和 *PGCH* 的面積分別是 60 和 18，求 ΔCEF 的面積。



Let *ABCD* be a rectangle. Let *E*, *F*, *G* and *H* be points on sides *DA*, *AB*, *BC* and *CD* respectively such that segments *EG* and *FH* divide the rectangle into four smaller rectangles. Let *P* be the intersection of *EG* and *FH*. If the areas of *ABCD* and *PGCH* are 60 and 18 respectively, find the area of ΔCEF.

6. 老師在黑板上寫下一個正實數 x，然後陳同學寫下 $5x$，李同學寫下 $8x$ 與 $\frac{1}{8x}$ 兩者中較大的一個，張同學則寫下 $25x$ 與 $\frac{1}{25x}$ 兩者中較大的一個。已知陳同學、李同學和張同學寫下的三個數之積為 1，求 x 的所有可能值之積。

The teacher wrote a positive real number x on the blackboard. Ann wrote down the value of $5x$, Ben wrote down $8x$ or $\frac{1}{8x}$ whichever is larger, while Cat wrote down $25x$ or $\frac{1}{25x}$ whichever is larger. It turns out that the product of the numbers written by Ann, Ben and Cat is equal to 1. Find the product of all possible values of x.

7. 設 n 為合成數，且 n 的每個正因數（除 n 本身外）均為一位數。求 n 的最大可能值。

Let n be a composite number such that every positive factor of n (other than n itself) is a one-digit number. Find the greatest possible value of n.

8. 已知從無窮數列 $a_1, a_2, a_3, \ldots$ 中，對任何正整數 n 皆有 $a_{n+1} = \frac{1}{1-a_n}$。若 k 是一個三位數，且 a_1 和 a_k 均等於 k，求 k 的最大可能值。

It is known that in the infinite sequence $a_1, a_2, a_3, \ldots$, we have $a_{n+1} = \frac{1}{1-a_n}$ for any positive integer n. If k is a three-digit number, and both a_1 and a_k are equal to k, find the greatest possible value of k.

第 9 至第 12 題，每題 5 分。

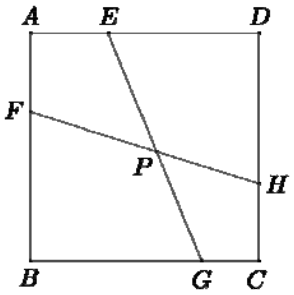
Questions 9 to 12 each carries 5 marks.

9. 設 a、b 和 c 為正整數，滿足 $a^2 = 2b^3 = 3c^5$。求 a 的最小可能值。

   Let a, b and c be positive integers satisfying $a^2 = 2b^3 = 3c^5$. Find the smallest possible value of a.

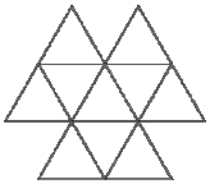
10. 設 $ABCD$ 為正方形，設 E、F、G 和 H 分別為邊 DA、AB、BC 和 CD 上的點，假設 P 是 EG 和 FH 的交點。若 $AB = 10$、$AF = CH = 4$、$BG = 9$ 及 $DE = 6$，求 $AFPE$ 的面積。

    Let $ABCD$ be a square. Let E, F, G and H be points on sides DA, AB, BC and CD respectively. Suppose P is the intersection of EG and FH. If $AB = 10$, $AF = CH = 4$, $BG = 9$ and $DE = 6$, find the area of $AFPE$.



11. 考慮如圖所示由 10 個等邊三角形組成的形狀。現將其中一些等邊三角形塗上綠色（可不把任何三角形塗色，或把全部三角形塗色），然後在每個綠色三角形中寫下與這個三角形相鄰的綠色三角形的數目（若兩個三角形其中一邊重合，則視兩者為相鄰；一個三角形並不與自己相鄰），再計算形狀中寫下的所有數之和（若沒有寫下任何數，總和則視為 0）。那麼，有多少種塗色的方法，可使得這個和是一個偶數？（不允許將形狀旋轉。）

    Consider the shape formed by 10 equilateral triangles as shown. Now some (possibly none or all) of the equilateral triangles are coloured green. Next, in each of the green triangles, one writes down the number of green triangles adjacent to this triangle. (Two triangles are adjacent if they share a common side. A triangle is not adjacent to itself.) The sum of all numbers written down in the shape is then computed. (The sum is regarded as 0 if no number is written down.) How many ways of colouring are there such that the sum is an even number? (Rotation of the shape is not allowed.)



12. 某校有 2018 名學生，他們的學生編號分別是 1 至 2018。他們被分成若干班，使得對任意質數 p 和正整數 a，編號為 a 和 pa 的學生都不會同班。那麼，該校最少有多少班？

There are 2018 students in a school, numbered 1 to 2018. They are divided into several classes in a way such that for any prime number p and positive integer a, the students with numbers a and pa will not belong to the same class. What is the minimum number of classes in the school?

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 對於正整數 n，設 $\omega(n)$ 代表 n 的不同質因數的數目。例如，由於 $40=2^3\times5$ 的質因數只有 2 和 5，我們有 $\omega(40)=2$。已知 $\omega(1)+\omega(2)+\cdots+\omega(1000)=2126$，求

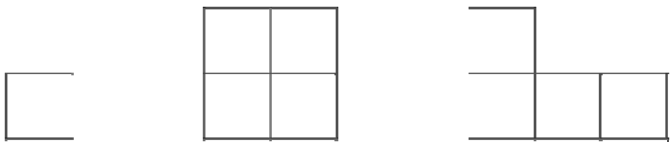
$$\omega(1)+\omega(1+2)+\omega(1+2+3)+\cdots+\omega(1+2+\cdots+1000)$$

的值。

Let $\omega(n)$ denote the number of distinct prime factors of a positive integer n. For example, since the only prime factors of $40=2^3\times5$ are 2 and 5, we have $\omega(40)=2$. Given that $\omega(1)+\omega(2)+\cdots+\omega(1000)=2126$, find the value of

$$\omega(1)+\omega(1+2)+\omega(1+2+3)+\cdots+\omega(1+2+\cdots+1000).$$

14. 我們須用 7 塊咭片完整地覆蓋一個 5×5 的方格表。這 7 塊咭片中，其中一塊是一個 1×1 的正方形，其餘六塊則是 2×2 的正方形或由 4 個方格組成的 L 形（見下圖；使用 L 形時可把它們反射和旋轉）。對於 5×5 方格表的一個方格，若存在一種覆蓋的方法使得 1×1 正方形被放在該個方格上，則稱它為「好格」。那麼共有多少個「好格」？



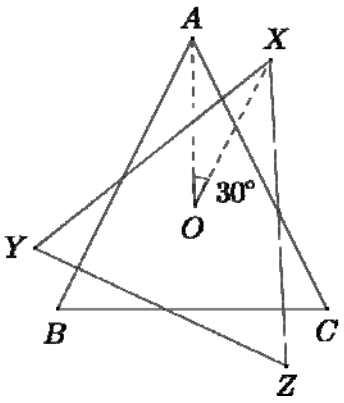
A 5×5 table is to be tiled using 7 cardboards, one of which is a 1×1 square while each of the other six is either a 2×2 square or an L-shape formed by 4 cells (see figure above; reflection and rotation of the L-shape are allowed). A cell of the 5×5 table is said to be 'good' if there exists a tiling such that the 1×1 square is put on that cell. How many 'good' cells are there?

15. 有 2018 名小童圍成一圈坐下，開始時部分人在唱歌（可能是全部人或沒有人）。在每分鐘的開始時，對於每一名小童，若與他相鄰的兩人在上一分鐘均在唱歌，那麼這名小童就會唱歌，否則他就不會唱歌。在 10 天後的某分鐘，有 n 名小童在唱歌。求 n 的所有可能值之和。

There are 2018 kids sitting in a circle. Initially, some of them (possibly all or none) are singing. At the beginning of every minute, a kid will sing if both of his neighbours were singing in the previous minute. Otherwise, the kid will not sing. After 10 days, n kids are singing in a certain minute. Find the sum of all possible values of n.

16. 設 ΔABC 為邊長為 10 的等邊三角形，並以 ΔABC 的中心為旋轉中心將三角形順時針旋轉 30° 得到 ΔXYZ。求 ΔABC 和 ΔXYZ 重疊部分的面積。

Let ΔABC be an equilateral triangle with side length 10. Using the centre of ΔABC as rotational centre, rotate the triangle clockwise by 30° to obtain ΔXYZ. Find the area of the overlapping region of ΔABC and ΔXYZ.



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 考慮座標平面上 36 個點 (x, y)，其中 x 和 y 是介乎 0 至 5 之間的整數。有多少種方法從這些點中選出 3 個點，使得它們組成一個面積為 10 的三角形？

Consider the 36 points (x, y) on the coordinate plane, where x and y are integers between 0 and 5. How many ways are there to choose 3 points among these points which form a triangle with area 10?

18. 姍姍報讀了一個數學課程，當中共有 7 次測驗。每次測驗的分數都必定是 –2、–1、1 或 2 分。每次測驗後，姍姍都會計算自己的累積分數。若 7 次測驗後的 7 個累積分數都是正數而且互不相同，則姍姍在該 7 次測驗的得分有多少組不同的可能性？

Sandy has enrolled in a mathematics course in which there are 7 tests. The score in each test must be one of –2, –1, 1 and 2. After each test, Sandy will compute her accumulated score. If the 7 accumulated scores after the 7 tests are all positive and pairwise distinct, how many different sets of possible scores are there for Sandy in the 7 tests?

19. 設 n 為一個三位數，使得 n^n 的最後三位數字是 237。求 n 的值。

Let n be a three-digit number such that the last three digits of n^n are 237. Find the value of n.

20. 在課室裏，老師對五名學生小陳、小李、小張、小王和小何說：「我寫下了一個數位兩兩不同的五位數 n。我會讓小陳看 n 的萬位和千位，讓小李看 n 的千位和百位，讓小張看 n 的百位和十位，讓小王看 n 的十位和個位，並讓小何看 n 的個位和萬位。」之後老師如所述般讓每名學生知道 n 的兩個數字，然後各人圍圈而坐，展開了以下的對話。

老師：「知道 n 的一個質因數的同學請舉手。」小王和小何舉手。

老師：「知道 n 的一個質因數的同學請舉手。」小陳、小王和小何舉手。

老師：「知道 n 的一個合成數因數的同學請舉手。」其中兩個同學舉手。

其中一個同學說：「那麼我知道 n 的值了，它是 7 的倍數。」

假設學生們都是聰明的，能按照其他學生的舉動作出正確分析。求 n 的值。

In a classroom, the teacher said to five students, Ann, Ben, Cat, Dan and Eva, 'I have written down a five-digit number n whose digits are pairwise distinct. I will let Ann see the ten thousands and thousands digits of n, let Ben see the thousands and hundreds digits of n, let Cat see the hundreds and tens digits of n, let Dan see the tens and unit digits of n and let Eva see the unit and ten thousands digits of n.' The teacher then let each student know two digits of n as said, and then everybody sat in a circle and started the following conversation.

'For those who know a prime factor of n, raise your hand,' the teacher said. Dan and Eva raised their hands.

'For those who know a prime factor of n, raise your hand,' the teacher said. Ann, Dan and Eva raised their hands.

'For those who know a composite factor of n, raise your hand,' the teacher said. Two of the students raised their hands.

'Then I know the value of n. It is a multiple of 7,' one of the students said.

Suppose the students are clever enough to analyse correctly what the other students have done. Find the value of n.

全卷完

END OF PAPER