第十六屆培正數學邀請賽（**2017** 年）

16th Pui Ching Invitational Mathematics Competition (2017)

決賽（高中組）

Final Event (Senior Secondary)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 現有兩隻電子錶。每當實際時間過了 1 小時，第一隻錶會走慢 3 分鐘，第二隻錶則會走快 3 分鐘。兩隻錶在每小時的開始時均會發出「嗶」聲。在正午時，兩隻錶被調整至正確時間。那麼，經過多少小時後，兩隻錶才會再同時發出「嗶」聲？

   There are two digital watches. The first watch runs 3 minutes slower than the actual time per actual hour, while the second watch runs 3 minutes faster than the actual time per actual hour. Both watches give a ‘beep’ sound at the beginning of every hour. At noon, the watches are adjusted to the correct time. How many hours will pass before both watches give ‘beep’ sounds at the same time again?

2. 有多少種方法在一個 2×3 的表格的格子中分別填上 1 至 6 這 6 個整數，使得每列和每行中的整數均按左至右和上至下的次序遞增？

   In how many ways can the cells of a 2×3 table be filled with the 6 integers from 1 to 6 such that the numbers in each row and column follow ascending order, from left to right and from top to bottom?

3. 某圖書館逢星期日及在「日」是 5 的倍數的日子均會休館。在某個月中，該圖書館最多會休館多少天？

   A library is closed every Sunday and whenever the ‘day’ in the date is a multiple of 5. What is the maximum number of days the library closes in a certain month?

4. 設 n 為正整數。若 n 可被 2、3、5 和 7 整除，但 n 的每位數字均不是 2、3、5 或 7，求 n 的最小可能值。

   Let n be a positive integer which is divisible by 2, 3, 5 and 7. If none of the digits of n is 2, 3, 5 or 7, find the smallest possible value of n.

第 5 至第 8 題，每題 4 分。

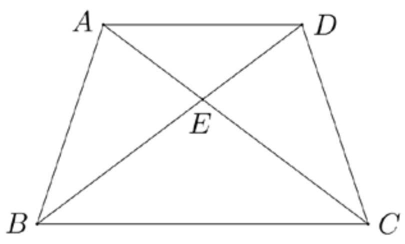
Questions 5 to 8 each carries 4 marks.

5. 若三次方程 $x^3-10x^2+mx-30=0$ 的所有根皆為正整數，求 m 的值。

   If all roots to the cubic equation $x^3-10x^2+mx-30=0$ are positive integers, find the value of m.

6. 圖中，$ABCD$ 是等腰梯形，其中 AD 與 BC 平行。E 是 AC 和 BD 的交點，且 $AE=3$、$BE=5$ 及 $\angle AEB=60°$。求 $ABCD$ 的面積。

   In the figure, $ABCD$ is an isosceles trapezium in which AD is parallel to BC. E is the intersection of AC and BD with $AE=3$, $BE=5$ and $\angle AEB=60°$. Find the area of $ABCD$.



7. 在數列 3, 6, 12, 14, 18, … 中，從第二項起每項均等於它的前一項加上其個位數（因此 18 之後的一項為 $18+8=26$）。若數列的第 k 項最接近 2017，求 k 的值。

   In the sequence 3, 6, 12, 14, 18, …, each term from the second term onwards is equal to the previous term added to its unit digit (hence the term after 18 is $18+8=26$). If the k-th term of the sequence is closest to 2017, find the value of k.

8. 方程 $(x-2017)^{2017}+(x-2016)^{2017}+\cdots+(x-1)^{2017}+x^{2017}+(x+1)^{2017}+\cdots+(x+2017)^{2017}=0$ 有多少個實數解？

   How many real number solutions are there to the following equation?

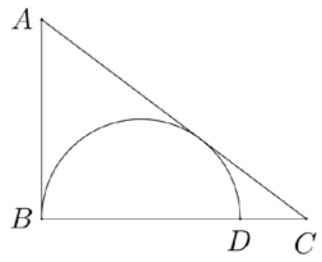
$$(x-2017)^{2017}+(x-2016)^{2017}+\cdots+(x-1)^{2017}+x^{2017}+(x+1)^{2017}+\cdots+(x+2017)^{2017}=0$$

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

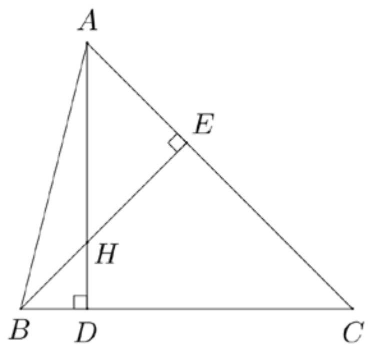
9. 圖中，ABC 是直角三角形，其中 B 是直角。D 是 BC 上的一點，使得 $BD=2CD=10$。若以 BD 為直徑的半圓與 AC 相切，求 AB 的長度。

   In the figure, ΔABC is right-angled at B. D is a point on BC such that $BD=2CD=10$. If the semi-circle with diameter BD is tangent to AC, find the length of AB.



10. AD 和 BE 為 ΔABC 的高，兩者相交於 H。若 $AB=16$、$AC=17$ 及 $BC=15$，求 DH 的長度。

    AD and BE are altitudes of ΔABC and they intersect at H. If $AB=16$, $AC=17$ and $BC=15$, find the length of DH.



11. 若 $2\sin x^\circ=\sqrt{2+\sqrt{2}}$，其中 $0<x<90$，求 x 的值。

    If $2\sin x^\circ=\sqrt{2+\sqrt{2}}$ where $0<x<90$, find the value of x.

12. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 ABCDE 所代表的五位數。

    In the multiplication shown, each letter represents a different digit from 0 to 9. Find the five-digit number represented by ABCDE.

			A	B	B	C
×					B	C
	D	D	E	E	A	C

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 若把合成數數列 4, 6, 8, 9, 10, … 的首 2^{1001} 項加起來，所得的和是個 n 位數。已知 $\log 2 \approx 0.30103$，求 n 的值。

    By adding up the first 2^{1001} terms of the sequence 4, 6, 8, 9, 10, … of composite numbers, the sum obtained is an n-digit number. Given $\log 2 \approx 0.30103$, find the value of n.

14. 已知 $\int \frac{1}{\sin x}\,dx = \ln\left|\frac{1}{\sin x} - \frac{1}{\tan x}\right| + C$，求 $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx$ 的值。

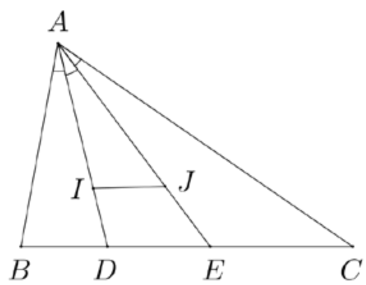
    Given that $\int \frac{1}{\sin x}\,dx = \ln\left|\frac{1}{\sin x} - \frac{1}{\tan x}\right| + C$, evaluate $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx$.

15. 某乒乓球比賽有 n 人參加，每名參賽者均與其餘各人對賽一次，而且沒有和局。小欣打算比賽結束後向其中三名參賽者索取簽名，當中其中一人必須在比賽中擊敗另外兩人。小欣發現，不論賽果如何，選出這三名參賽者均有最少 10395 種方法。求 n 的最小可能值。

    There are n participants in a table tennis tournament. Each participant plays one game against every other participant, and there is no tie in each game. Yanni plans to obtain the autographs of three players after the tournament, with the condition that one of them must have beaten the other two during the tournament. Yanni found that, regardless of the outcome of the games, she has at least 10395 ways to choose these three players. Find the smallest possible value of n.

16. 在 ΔABC 中，$AB = 35$、$BC = 117$ 及 $CA = 100$。D 和 E 為 BC 上的點，使得 $\angle BAD = \angle DAE = \angle EAC$。若 I 和 J 分別為 ΔABE 和 ΔADC 的內心，求 IJ^2 的值。

    In ΔABC, $AB = 35$, $BC = 117$ and $CA = 100$. D and E are points on BC such that $\angle BAD = \angle DAE = \angle EAC$. If I and J are the incentres of ΔABE and ΔADC respectively, find the value of IJ^2.



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 m、n 為正整數，使得 $125(m^2+n^2)=(m+n)^3$。求 m 的所有可能值之和。

Let m, n be positive integers such that $125(m^2+n^2)=(m+n)^3$. Find the sum of all possible values of m.

18. 設 a、b、c 為實數。若 $12(a^2+2b^2+3c^2)=(a+2b)^2+(2b+3c)^2+(a+3c)^2+4b(a+2b+3c)-a^2-9c^2$，且 $a+b=2016$，求 c 的值。

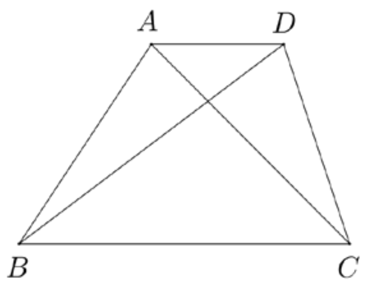
Let a, b, c be real numbers. If $12(a^2+2b^2+3c^2)=(a+2b)^2+(2b+3c)^2+(a+3c)^2+4b(a+2b+3c)-a^2-9c^2$ and $a+b=2016$, find the value of c.

19. 桌子上有 4 張正面朝下的咭片，現每次隨機選一張咭片並將其翻轉，直至所有咭片均是正面朝上為止。以 n 表示把咭片翻轉的總次數。求 n 是 4 的倍數的概率。

There are 4 cards on the table, initially all facing downwards. Each time one of the cards is randomly chosen and flipped, until all cards are facing upwards. Let n denote the total number of flippings. Find the probability that n is a multiple of 4.

20. 在四邊形 $ABCD$ 中，$\angle BAC=88.88°$、$\angle CAD=46.68°$、$\angle ADB=30°$、$\angle BDC=74.44°$ 及 $\angle ACB=x°$。求 x 的值。

In quadrilateral $ABCD$, $\angle BAC=88.88°$, $\angle CAD=46.68°$, $\angle ADB=30°$, $\angle BDC=74.44°$ and $\angle ACB=x°$. Find the value of x.



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