第十六屆培正數學邀請賽（2017 年）

16th Pui Ching Invitational Mathematics Competition (2017)

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 某城市為控制交通流量，規定單數號碼的私家車只能在單數日子（即每月的 1、3、5、… 日）行駛，雙數號碼的私家車只能在雙數日子行駛。在 2017 年，單數號碼的私家車可以在多少天行駛？

   In an attempt to control traffic flow, a city enacts the rule that private cars with odd-numbered plates may only be used on odd-numbered dates (i.e. the 1st, 3rd, 5th… days of each month) while those with even-numbered plates may only be used on even-numbered dates. On how many days may a private car with an odd-numbered plate be used in the year 2017?

2. 現有兩隻電子錶。每當實際時間過了 1 小時，第一隻錶會走慢 3 分鐘，第二隻錶則會走快 3 分鐘。兩隻錶在每小時的開始時均會發出「嗶」聲。在正午時，兩隻錶被調整至正確時間。那麼，經過多少小時後，兩隻錶才會再同時發出「嗶」聲？

   There are two digital watches. The first watch runs 3 minutes slower than the actual time per actual hour, while the second watch runs 3 minutes faster than the actual time per actual hour. Both watches give a ‘beep’ sound at the beginning of every hour. At noon, the watches are adjusted to the correct time. How many hours will pass before both watches give ‘beep’ sounds at the same time again?

3. 某等比數列共有 10 項，其中首項是 2，公比是 4。若 10 項之積為 2^k，求 k 的值。

   A geometric sequence consists of 10 terms. The first term is 2 and the common ratio is 4. If the product of the 10 terms is 2^k, find the value of k.

4. 方程 $(x^2-3x+4)^2=(3x-5)^2$ 有多少個不同實根？

   How many distinct real roots are there to the equation $(x^2-3x+4)^2=(3x-5)^2$?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

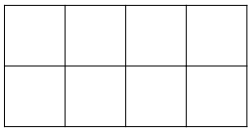
5. 若三次方程 $x^3-10x^2+mx-30=0$ 的所有根皆為正整數，求 m 的值。

   If all roots to the cubic equation $x^3-10x^2+mx-30=0$ are positive integers, find the value of m.

6. 設 p_1、p_2、…、p_{1000} 為質數。若 $p_1+p_2+\cdots+p_{1000}=3000$，求 p_1 的最大可能值。

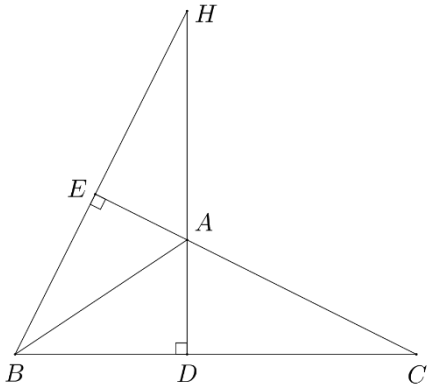
   Let p_1, p_2, …, p_{1000} be prime numbers. If $p_1+p_2+\cdots+p_{1000}=3000$, find the greatest possible value of p_1.

7. 有多少種方法在一個 2×4 的表格的格子中分別填上 1 至 8 這 8 個整數，使得每列和每行中的整數均按左至右和上至下的次序遞增？

   In how many ways can the cells of a 2×4 table be filled with the 8 integers from 1 to 8 such that the numbers in each row and column follow ascending order, from left to right and from top to bottom?

8. AD 和 BE 為 ΔABC 的高，且 H 是兩者的交點（如有需要可適當地延長）。若 $AB=16$、$AC=17$ 及 $CD=15$，求 DH 的長度。

   AD and BE are altitudes of ΔABC and H is their intersection (suitably produced if necessary). If $AB=16$, $AC=17$ and $CD=15$, find the length of DH.

第 9 至第 12 題，每題 5 分。

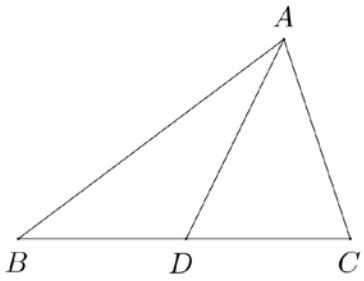
Questions 9 to 12 each carries 5 marks.

9. 有多少種方法可以從首 40 個正整數中選取 6 個數 a、b、c、d、e、f，使得 $a<b<c<d<e<f$ 且 $\log a+\log b+\log c+\log d+\log e+\log f$ 是整數？

   In how many ways can one choose 6 numbers a, b, c, d, e, f from the first 40 positive integers such that $a<b<c<d<e<f$ and $\log a+\log b+\log c+\log d+\log e+\log f$ is an integer?

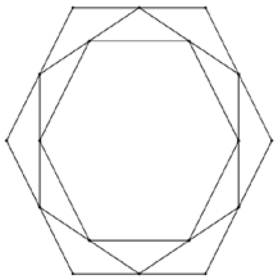
10. ΔABC 的周長為 110。設 D 為 BC 的中點。若 $AD = BD = CD = 25$，求 ΔABC 的面積。

ΔABC has perimeter 110. Let D be the midpoint of BC. If $AD = BD = CD = 25$, find the area of ΔABC.



11. 給定正六邊形 A，我們可將 A 的相鄰邊的中點連結起來，組成六邊形 B，同樣地，我們可將 B 的相鄰邊的中點連結起來，組成六邊形 C。若 A 的面積是 144，求 C 的面積。

Given a regular hexagon A, one can form a hexagon B by joining the midpoints of the consecutive sides of A. Similarly, one can form a hexagon C by joining the midpoints of the consecutive sides of B. If the area of A is 144, find the area of C.



12. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求 ABCDE 所代表的五位數。

In the multiplication shown, each letter represents a different digit from 0 to 9. Find the five-digit number represented by ABCDE.

$$\begin{array}{cccccc} & & A & B & B & C \\ \times & & & & B & C \\ \hline D & D & E & E & A & C \end{array}$$

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

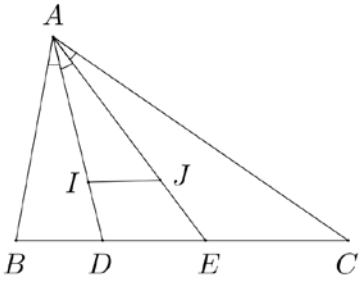
13. 一個電腦遊戲中共有四個按鈕，分別會使遊戲中的主角向上、下、左或右移動 1 個單位。先後四次隨機按下其中一個按鈕後，主角距離起點 x 個單位。求 x 的期望值。

There are four buttons in a computer game, which will move the character by 1 unit upward, downward, to the left and to the right respectively. In each round, a randomly-chosen button is pressed. After 4 rounds, the character ends at a distance of x units from the starting point. Find the expected value of x.

14. 若 $2\sin x° = \sqrt{2+\sqrt{2+\sqrt{2}}}$，其中 $0 < x < 90$，求 x 的值。

If $2\sin x° = \sqrt{2+\sqrt{2+\sqrt{2}}}$ where $0 < x < 90$, find the value of x.

15. 在 ΔABC 中，$AB=35$、$BC=117$ 及 $CA=100$。D 和 E 為 BC 上的點，使得 $\angle BAD=\angle DAE=\angle EAC$。若 I 和 J 分別為 ΔABE 和 ΔADC 的內心，求 IJ^2 的值。



In ΔABC, $AB=35$, $BC=117$ and $CA=100$. D and E are points on BC such that $\angle BAD=\angle DAE=\angle EAC$. If I and J are the incentres of ΔABE and ΔADC respectively, find the value of IJ^2.

16. 現要在算式「1□2□⋯□9」中的 8 個空格內都分別填入「+」、「−」或「×」，使其結果是一個偶數。共有多少種不同的方法填入符號？

Each of the 8 squares in the expression '1□2□⋯□9' is to be filled with one of the symbols '+', '–' or '×' so that the result is an even number. In how many different ways can this be done?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 a、b、c、d 為非零整數，使得 $3a^2+20ac+31c^2=-1$ 及 $3ab+10bc+10ad+31cd=0$。求 $3b^2+20bd+31d^2$ 的最小可能值。

Let a, b, c, d be nonzero integers such that $3a^2+20ac+31c^2=-1$ and $3ab+10bc+10ad+31cd=0$. Find the smallest possible value of $3b^2+20bd+31d^2$.

18. 設 a、b、c 為實數。若 $12(a^2+2b^2+3c^2)=(a+2b)^2+(2b+3c)^2+(a+3c)^2+4b(a+2b+3c)-a^2-9c^2$，且 $a+b=2016$，求 c 的值。

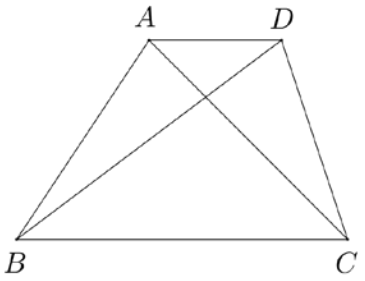
Let a, b, c be real numbers. If $12(a^2+2b^2+3c^2)=(a+2b)^2+(2b+3c)^2+(a+3c)^2+4b(a+2b+3c)-a^2-9c^2$ and $a+b=2016$, find the value of c.

19. 在一個遊戲中，老師暗中寫下一個一位正整數，然後小陳、小張和小王三位同學需順序在黑板上寫下一個一位正整數（同一整數只能寫一次），其中最接近老師的數字者（可多於一位）可獲 1 顆糖果。每位同學都使用以下策略，亦得知其他同學的策略相同：每人寫下整數時，均選一個整數使得自己可獲糖果數目的期望值最大，如這樣的整數超過一個，則會隨機選其中的一個數。設 a、b、c 分別表示小陳、小張和小王所選的數。求三位數 $\overline{abc}$ 的最大可能值。

In a game, the teacher secretly writes down a one-digit positive integer, and three students, Ann, Ben and Cat must then write down a one-digit positive integer on the board in order (the same integer can only be written once). The student(s) (possibly more than one) whose number is closest to the teacher's will get 1 candy. Each student adopts the following strategy, and they know that others will do the same. The strategy is to choose the integer which maximises the expected number of candies they could get, and, in case more than one such integer exists, randomly pick one of them. If a, b and c denote the integers written by Ann, Ben and Cat respectively, find the greatest possible value of the three-digit number $\overline{abc}$.

20. 在四邊形 $ABCD$ 中，$\angle BAC = 88.88°$、$\angle CAD = 46.68°$、$\angle ADB = 30°$、$\angle BDC = 74.44°$ 及 $\angle ACB = x°$。求 x 的值。

In quadrilateral $ABCD$, $\angle BAC = 88.88°$, $\angle CAD = 46.68°$, $\angle ADB = 30°$, $\angle BDC = 74.44°$ and $\angle ACB = x°$. Find the value of x.



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