第十五屆培正數學邀請賽（2016 年）

15th Pui Ching Invitational Mathematics Competition (2016)

決賽（高中組）

Final Event (Senior Secondary)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 已知 n 個連續正整數之和是 2016，其中 $2 \le n \le 10$。求 n 的所有可能值之和。

   The sum of n consecutive positive integers is 2016, where $2 \le n \le 10$. Find the sum of all possible values of n.

2. 在一個遊戲中，10 個紅球和 12 個藍球被放進一個袋子中，然後參加者每次抽出一個球，直至抽出所有球為止。從第二個球起，如果抽出的球的顏色和前一個抽出的球相同，便可得 1 分。若某位參加者抽出的第一個球和最後一個球都是紅色的，求他的得分的最小可能值。

   In a game, 10 red balls and 12 blue balls are put into a bag. The player then draws one ball each time until all balls are drawn. Starting from the second ball, 1 point will be awarded if the ball drawn has the same colour as the previous one. If a player draws a red ball in the first as well as the last draw, what is his minimum possible score?

3. 對任意實數 x，函數 f 皆滿足方程 $f\left(\frac{x}{3}\right) = x^2 + x + 1$。若 $f(3y) = 7$，求 y 的所有可能值之和。

   The function f satisfies $f\left(\frac{x}{3}\right) = x^2 + x + 1$ for any real number x. If $f(3y) = 7$, find the sum of all possible values of y.

4. 有多少個正整數 k 可使得 $\log_k 729$ 是正整數？

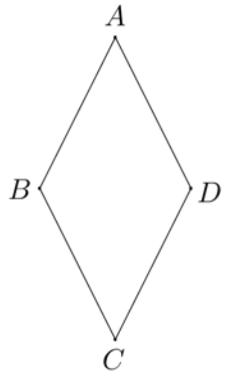
   How many positive integers k are there such that $\log_k 729$ is a positive integer?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

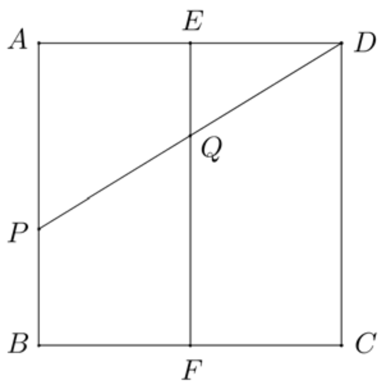
5. $ABCD$ 是邊長為 1 的菱形，其中 $\angle BAD < 90°$。若 $ABCD$ 的面積是 $\frac{3}{5}$，求 $\frac{AC}{BD}$ 的值。

   $ABCD$ is a rhombus of side length 1 with $\angle BAD < 90°$. If the area of $ABCD$ is $\frac{3}{5}$, find the value of $\frac{AC}{BD}$.



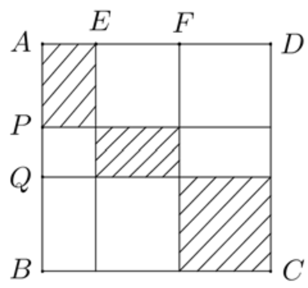
6. 圖中，$ABCD$ 是邊長為 8 的正方形，點 E 和 F 分別在邊 AD 和 BC 上，使得 EF 和 AB 平行。P 是邊 AB 上的點，使得 $AP = 6$。Q 是 PD 和 EF 的交點。若四邊形 $APQE$ 和 $CDQF$ 的面積相同，求 EQ 的長度。

   In the figure, $ABCD$ is a square of side length 8. Points E and F lie on sides AD and BC respectively such that EF and AB are parallel. P is a point on side AB with $AP = 6$. Q is the intersection of PD and EF. If quadrilaterals $APQE$ and $CDQF$ have the same area, find the length of EQ.



7. 圖中，$ABCD$ 是邊長為 10 的正方形。點 E 和 F 在邊 AD 上，P 和 Q 在邊 AB 上，使得 AE、EF、FD、AP、PQ、QB 的長度均不小於 2。分別穿過 E、F、P、Q 且與正方形邊平行的直線將正方形分成 9 個區域。求陰影部分總面積的最大可能值。

   In the figure, $ABCD$ is a square with side length 10. Points E and F lie on side AD, while P and Q lie on side AB, in such a way that AE, EF, FD, AP, PQ, QB have lengths at least 2. Lines through E, F, P, Q parallel to the sides of the square are drawn respectively and they divide the square into 9 regions. Find the largest possible total area of the shaded regions.



8. 設 $PABC$ 為三角錐體，其中 $PA = 1$、$PB = 2$、$PC = 6$，且 PA、PB、PC 當中任意兩者之間的夾角均為 60°。求 ΔABC 的面積。

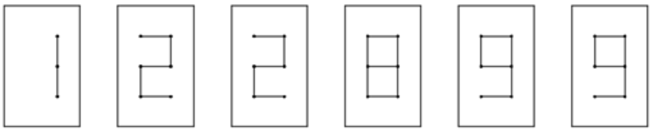
   Let $PABC$ be a triangular pyramid with $PA = 1$, $PB = 2$, $PC = 6$ and in which the angles between any two of PA, PB, PC are 60°. Find the area of ΔABC.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 利用以下 6 張印有電子數字的透明咭片共可組成多少個六位數？（註：透明咭片可倒轉或反轉使用。）

   How many six-digit numbers can be formed using the following 6 transparent cards with digital numbers? (Note: The transparent cards may be put upside down or turned over.)

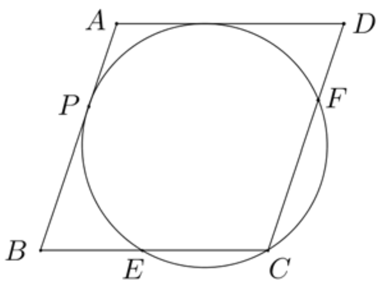


10. 一部特別的計算機上只有兩個分別是紅色和綠色的按鈕。當螢幕上的數是 n 時，按下紅色按鈕會令 n 變成 $n+r$，其中 r 是 n 除以 5 的餘數；按下綠色按鈕則會令 n 變成 $2n$（例如：當螢幕上顯示 18 時，按下紅色按鈕會使螢幕上的數變成 21，按下綠色按鈕則變成 36）。若開始時螢幕上的數是 1，那麼最少需按下多少次按鈕，才可令螢幕上的數變成 2016？

    On a special calculator there are only two buttons which are red and green respectively. When the number shown on screen is n, pressing the red button will turn n to $n+r$ where r is the remainder when n is divided by 5; pressing the green button will turn n to $2n$. (For instance, if 18 is shown on screen, pressing the red button makes it 21 while pressing the green button makes it 36.) If the initial number on the screen is 1, what is the minimum number of times the buttons need to be pressed in order to turn the number on screen to 2016?

11. 圖中，$ABCD$ 是平行四邊形。一個圓與邊 AD 相切，並與邊 AB 相切於 P，且該圓穿過 C 並分別交 BC 和 CD 於 E 和 F。若 $BE:EC=1:3$ 及 $DF:FC=1:8$，求 $\frac{AP}{PB}$ 的值。

    In the figure, $ABCD$ is a parallelogram. A circle is tangent to side AD and is tangent to side AB at P. Also, the circle passes through C and intersect BC and CD at E and F respectively. If $BE:EC=1:3$ and $DF:FC=1:8$, find the value of $\frac{AP}{PB}$.



12. 設 a 為整數，b 為實數。若方程 $x^3+ax^2+bx-2016^2=0$ 只有一個實根 $x=2016$，求 a 的最小可能值。

Let a be an integer and b be a real number. If the equation $x^3+ax^2+bx-2016^2=0$ has only one real root $x=2016$, find the smallest possible value of a.

第 13 至第 16 題，每題 6 分。

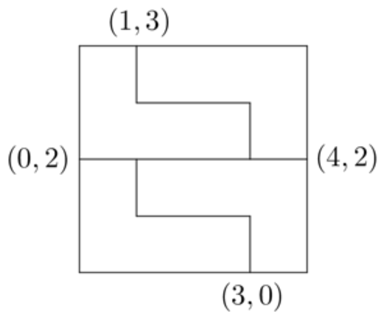
Questions 13 to 16 each carries 6 marks.

13. 在座標平面上，某個正 2017 邊形以原點為中心。若其中一個頂點的座標是 $(1,0)$，求另外 2016 個頂點的所有 x 座標和 y 座標之和。

On the coordinate plane, a regular 2017-sided polygon is centred at the origin. If one of the vertices has coordinates $(1,0)$, find the sum of the x-coordinates and the y-coordinates of all other 2016 vertices.

14. 給定座標平面上以 $(0,0)$、$(4,0)$、$(4,4)$ 和 $(0,4)$ 為頂點的正方形，現要加上一些線段以把正方形分割成 4 個相同的連通區域（即每個區域均可通過平移和旋轉得到另一個區域），其中每條線段必須與正方形的一邊平行，且其端點的座標必須為整數（圖中給出其中一種分割正方形的方法）。那麼有多少種方法加上線段？

A square on the coordinate plane with vertices $(0,0)$, $(4,0)$, $(4,4)$ and $(0,4)$ is given. Now line segments are to be drawn to divide the square into 4 identical connected regions (i.e. each region can be transformed to another by translation and rotation), with the restriction that each segment must be parallel to a side of the square and its endpoints must have integer coordinates. (The figure shows one possible way to divide the square.) How many different ways are there to draw the line segments?



15. 現有 100 張咭片，分別寫上 00、01、…、99。若隨機抽出兩張，則在兩張咭片上分別出現同一個數字的概率是多少？

There are 100 cards, with 00, 01, …, 99 written on them respectively. If two cards are chosen at random, what is the probability that there is a same digit appearing on both cards?

16. 三人進行遊戲如下：他們把九條長度分別為 1、2、⋯、9 的木棒隨機分配，使得每人有三條木棒，然後各自嘗試以所得的木棒組成三角形，並以所組成三角形面積最大者為勝。（若沒有人能組成三角形，則無人勝出。若出現和局，則以抽籤定勝負。）勝出者所組成的三角形的面積的最小可能值是多少？

Three people play a game as follows. They randomly distribute nine sticks with lengths 1, 2, …, 9 respectively among themselves so that each has three sticks. Each player then tries to form a triangle with the three sticks obtained and the one whose triangle has the greatest area wins. (If nobody can form a triangle, there is no winner. If there is a tie, the winner is determined by drawing lots.) What is the smallest possible area of the triangle formed by the winner?

第 17 至第 20 題，每題 7 分。

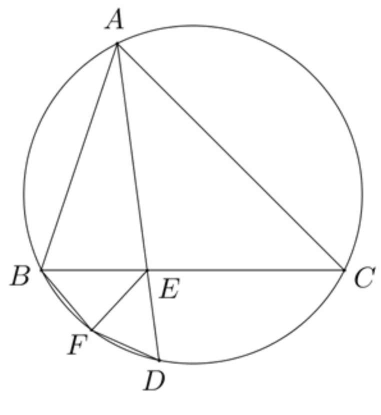
Questions 17 to 20 each carries 7 marks.

17. 在一個 5×5 方格表的每格內各有 10 枚金幣。在每一步中，我們可以選擇同一行或同一列 4 個相連且各有最少一枚金幣的方格，然後從這 4 個方格內各取走一枚金幣。按此規則，經過若干步後最多可取走多少枚金幣？

In each cell of a 5×5 grid, there are 10 coins. In each step, one can choose 4 consecutive cells in a row or in a column, each of which contains at least one coin, and then take away a coin from each of these 4 cells. At most how many coins can be taken away after some steps according to such regulation?

18. 在 ΔABC 中，$\angle BAC$ 的內角平分線分別與邊 BC 和 ΔABC 的外接圓交於 E 和 D，且 ΔBDE 的外心 F 位於 ΔABC 的外接圓上。若 ΔABC 的外接圓半徑是 ΔBDE 的外接圓半徑的 3 倍，求 $\cos\angle ABC$ 的值。

In ΔABC, the internal bisector of $\angle BAC$ meets side BC and the circumcircle of ΔABC at E and D respectively, and the circumcentre F of ΔBDE lies on the circumcircle of ΔABC. If the circumradius of ΔABC is 3 times the circumradius of ΔBDE, find the value of $\cos\angle ABC$.



19. 若在 $-5 \le a \le 5$ 和 $-3 \le b \le 3$ 的約束條件下隨機選兩個實數 a、b，則方程組 $\begin{cases} x^2 = b^2 + 2ax + 4a - 4 \\ y^2 = b^2 - 2ay - 4a - 4 \end{cases}$ 有實數解的概率是多少？

If real numbers a and b are randomly chosen subject to the constraints $-5 \le a \le 5$ and $-3 \le b \le 3$, what is the probability that the system of equations $\begin{cases} x^2 = b^2 + 2ax + 4a - 4 \\ y^2 = b^2 - 2ay - 4a - 4 \end{cases}$ has a real solution?

20. 設 $f(x,y)$ 和 $g(x,y)$ 為兩個未知數的多項式。若對任意滿足 $x^2y + 6y = xy^3 + 5x^2 + 2x$ 的實數 x 和 y，皆有 $(x^2 + 2xy + 3y^2)f(x,y) = (4x^2 + 5xy + 6y^2)g(x,y)$，且 $g(0,0) = 6$，求 $f(0,0)$ 的值。

Let $f(x,y)$ and $g(x,y)$ be polynomials in two variables. Suppose $g(0,0) = 6$, and that $(x^2 + 2xy + 3y^2)f(x,y) = (4x^2 + 5xy + 6y^2)g(x,y)$ for any real numbers x and y satisfying $x^2y + 6y = xy^3 + 5x^2 + 2x$. Find the value of $f(0,0)$.

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