第十五屆培正數學邀請賽（2016 年）

15th Pui Ching Invitational Mathematics Competition (2016)

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 在一個遊戲中，10 個紅球和 10 個藍球被放進一個袋子中，然後參加者每次抽出一個球，直至抽出所有球為止。從第二個球起，如果抽出的球的顏色和前一個抽出的球相同，便可得 1 分。若某位參加者抽出的第一個球和最後一個球都是紅色的，求他的得分的最小可能值。

   In a game, 10 red balls and 10 blue balls are put into a bag. The player then draws one ball each time until all balls are drawn. Starting from the second ball, 1 point will be awarded if the ball drawn has the same colour as the previous one. If a player draws a red ball in the first as well as the last draw, what is his minimum possible score?

2. 對任意實數 x，函數 f 皆滿足方程 $f\left(\frac{x}{3}\right)=x^2+x+1$。若 $f(3y)=7$，求 y 的所有可能值之和。

   The function f satisfies $f\left(\frac{x}{3}\right)=x^2+x+1$ for any real number x. If $f(3y)=7$, find the sum of all possible values of y.

3. 求 $\log_6\frac{3}{1}+\log_6\frac{4}{2}+\log_6\frac{5}{3}+\cdots+\log_6\frac{9}{7}$ 的值。

   Find the value of $\log_6\frac{3}{1}+\log_6\frac{4}{2}+\log_6\frac{5}{3}+\cdots+\log_6\frac{9}{7}$.

4. 有多少個正整數 k 可使得 $\log_k 729$ 是正整數？

   How many positive integers k are there such that $\log_k 729$ is a positive integer?

第 5 至第 8 題，每題 4 分。

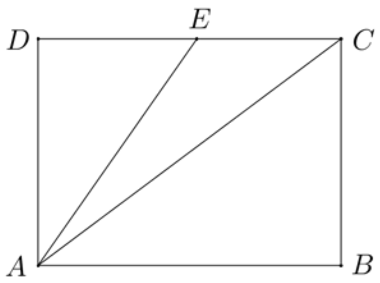
Questions 5 to 8 each carries 4 marks.

5. 一個正方體的體積是 36。一個圓錐體的高與該正方體相同，且其底部的直徑等於正方體的邊長。求該圓錐體的體積。

   The volume of a cube is 36. A cone has the same height as the cube, and its base diameter is the same as the side length of the cube. Find the volume of the cone.

6. 設 $ABCD$ 為長方形，其中 $AD = 3$。點 E 在邊 CD 上，使得 AC 和 AE 三等分 $\angle DAB$。求 ΔACE 的周長。

   Let $ABCD$ be a rectangle with $AD = 3$. E is a point on side CD such that AC and AE trisect $\angle DAB$. Find the perimeter of ΔACE.

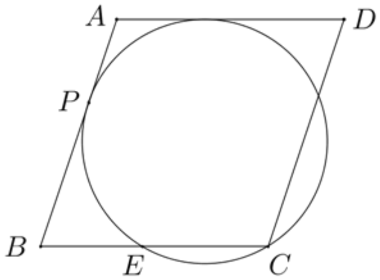


7. 現有 9 張咭片，分別寫上 00、01、02、10、11、12、20、21、22。若隨機抽出兩張，則在兩張咭片上分別出現同一個數字的概率是多少？

   There are 9 cards, with 00, 01, 02, 10, 11, 12, 20, 21, 22 written on them respectively. If two cards are chosen at random, what is the probability that there is a same digit appearing on both cards?

8. 圖中，$ABCD$ 為菱形。一個圓與邊 AD 相切，並與邊 AB 相切於 P，且該圓穿過 C 並交 BC 於 E。若 $BE : EC = 1 : 3$，求 $\frac{AP}{PB}$ 的值。

   In the figure, $ABCD$ is a rhombus. A circle is tangent to side AD and is tangent to side AB at P. Also, the circle passes through C and intersects BC at E. If $BE : EC = 1 : 3$, find the value of $\frac{AP}{PB}$.



第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 設 x 為實數。若 $\frac{1}{x+2}$、$\frac{x}{x-1}$、$1-\frac{1}{4x+1}$ 其中一項是另外兩項之積，問 x 有多少個可能值？

   Let x be a real number. If one of $\frac{1}{x+2}$, $\frac{x}{x-1}$, $1-\frac{1}{4x+1}$ is the product of the other two, how many possible values of x are there?

10. 設 a 為整數，b 為實數。若方程 $x^3+ax^2+bx-25=0$ 只有一個實根 $x=5$，求 a 的所有可能值之和。

    Let a be an integer and b be a real number. If the equation $x^3+ax^2+bx-25=0$ has only one real root $x=5$, find the sum of all possible values of a.

11. 某城市使用一套特別的系統表示整數，當中有 3 種不同的數碼 A、B、C，其數值分別對應於我們平常使用的十進制中的 1、6、36。該系統中的 n 位數形如 $\overline{\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n}$，其中每個 X_i 都是 A、B、C 之一，而它等於十進制中的 $\mathrm{X}_1\pm\mathrm{X}_2\pm\cdots\pm\mathrm{X}_n$，當中若 $\mathrm{X}_i>\mathrm{X}_{i+1}$，則 X_i 和 X_{i+1} 之間取負號，否則取正號。例如，該系統中的四位數 $\overline{\mathrm{ACCB}}$ 等於十進制中的 $1+36+36-6=67$。在這個系統中的三位數共可表示多少個不同的值？

    A city uses a special system to represent integers. In the system, there are 3 different numerals A, B, C, corresponding to the values 1, 6, 36 respectively in our usual decimal system. An n-digit number in this system is of the form $\overline{\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n}$, where each X_i is one of A, B, C, and it is equivalent to the number $\mathrm{X}_1\pm\mathrm{X}_2\pm\cdots\pm\mathrm{X}_n$ in the decimal system, where the sign between each X_i and X_{i+1} is negative if $\mathrm{X}_i>\mathrm{X}_{i+1}$, and is positive otherwise. For example, the four-digit number $\overline{\mathrm{ACCB}}$ in this system is equal to $1+36+36-6=67$ in the decimal system. How many different values can be represented by the three-digit numbers in this system?

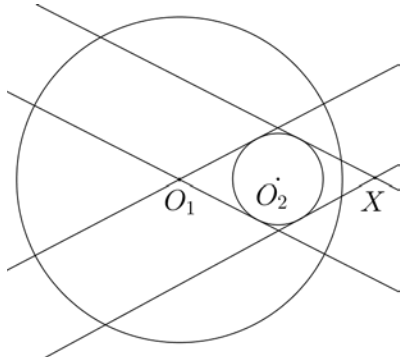
12. 某袋子中裝有 99 個球，分別編號為 01、02、⋯、99。現要從中選出五個球並把它們的編號作為「幸運號碼」，且須滿足以下條件：把每個「幸運號碼」的兩位數字左右倒轉後，所得的仍是一個「幸運號碼」。那麼，選球的方法共有多少種？

    There are 99 balls in a bag, numbered 01, 02, …, 99 respectively. Now 5 balls are to be chosen and their numbers be designated as ‘lucky numbers’, with the requirement that by swapping the two digits of each ‘lucky number’, the result must still be a ‘lucky number’. How many ways are there to choose the balls?

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 圖中，大圓和小圓的半徑分別為 5 和 1，圓心分別為 O_1 和 O_2，其中 $O_1O_2=3$。現從 O_1 向小圓作兩條切線，再作小圓的另外兩條切線，分別與前述的兩條切線之一平行，且後兩條切線相交於 X。求 X 對大圓切線和 X 對小圓切線的長度之差。



In the figure, the larger and smaller circles have radii 5 and 1 respectively and centres O_1 and O_2 respectively, with $O_1O_2=3$. From O_1, two tangents to the smaller circle are constructed. Two other tangents to the smaller circle, each parallel to one of the two aforementioned tangents, are then constructed, and these two tangents meet at X. Find the difference in length between a tangent from X to the smaller circle and a tangent from X to the larger circle.

14. 一個遊戲共設 n 個回合，並在地上排成一行的 $n+1$ 個正方形上進行，其中 n 是正整數。遊戲開始時，參加者站在最左邊的正方形上，並放下一個球。在第 k 個回合，參加者須走 k 步，每步均須向左或向右走到一個相鄰的正方形（同一回合中可向不同方向走），完成後在最後的位置放下一個球。如果在 n 個回合後，所有正方形內都放有球，參加者便勝出。已知在遊戲中，參加者共須走至少 2016 步，且任何情況下均不可能勝出。求 n 的最小可能值。

A game with n rounds is played on $n+1$ squares arranged in a row on the ground, where n is a positive integer. Initially, the player stands on the leftmost square and puts down a ball. In the k-th round, the player moves k times, each leftward or rightward to an adjacent square (moving towards different directions within the same round is allowed), and puts down a ball in the final position. If every square contains a ball after n rounds, the player wins. It is given that the player needs to move at least 2016 times in the game, and will never be able to win. Find the smallest possible value of n.

15. 三人進行遊戲如下：他們把九條長度分別為 1、2、⋯、9 的木棒隨機分配，使得每人有三條木棒，然後各自嘗試以所得的木棒組成三角形，並以所組成三角形面積最大者為勝。（若沒有人能組成三角形，則無人勝出。若出現和局，則以抽籤定勝負。）勝出者所組成的三角形的面積的最小可能值是多少？

Three people play a game as follows. They randomly distribute nine sticks with lengths 1, 2, …, 9 respectively among themselves so that each has three sticks. Each player then tries to form a triangle with the three sticks obtained and the one whose triangle has the greatest area wins. (If nobody can form a triangle, there is no winner. If there is a tie, the winner is determined by drawing lots.) What is the smallest possible area of the triangle formed by the winner?

16. 對正整數 n，設 $f(n)$ 表示 $\sqrt{2n+\frac{1}{4}}$ 四捨五入至最接近整數後所得的值，例如：當 $n=4$ 時，$\sqrt{2n+\frac{1}{4}}=\sqrt{8.25}\approx 2.9$，故此 $f(4)=3$。求 $f(1)+f(2)+\cdots+f(2016)$ 的值。

For positive integer n, let $f(n)$ denote the result of rounding off $\sqrt{2n+\frac{1}{4}}$ to the nearest integer. For example, when $n=4$, we have $\sqrt{2n+\frac{1}{4}}=\sqrt{8.25}\approx 2.9$ and so $f(4)=3$. Find the value of $f(1)+f(2)+\cdots+f(2016)$.

第 17 至第 20 題，每題 7 分。

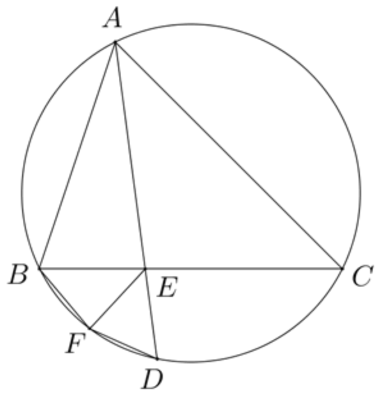
Questions 17 to 20 each carries 7 marks.

17. 一個正整數 n 以 a 進制表示時是一個三位數字皆相同的三位數，以 b 進制表示時是一個四位數字皆相同的四位數。求 n 的最小可能值（以十進制表示）。

A positive integer n is a three-digit number composed of the same digit in base a representation, and is a four-digit number composed of the same digit in base b representation. Find (in base 10) the smallest possible value of n.

18. 在 ΔABC 中，$\angle BAC$ 的內角平分線分別與邊 BC 和 ΔABC 的外接圓交於 E 和 D，且 ΔBDE 的外心 F 位於 ΔABC 的外接圓上。若 ΔABC 的外接圓半徑是 ΔBDE 的外接圓半徑的 3 倍，求 $\cos\angle ABC$ 的值。

In ΔABC, the internal bisector of $\angle BAC$ meets side BC and the circumcircle of ΔABC at E and D respectively, and the circumcentre F of ΔBDE lies on the circumcircle of ΔABC. If the circumradius of ΔABC is 3 times the circumradius of ΔBDE, find the value of $\cos\angle ABC$.



19. 設 $f(x,y)$ 和 $g(x,y)$ 為兩個未知數的多項式。若對任意滿足 $x^2y+6y=xy^3+5x^2+2x$ 的實數 x 和 y，皆有 $(x^2+2xy+3y^2)f(x,y)=(4x^2+5xy+6y^2)g(x,y)$，且 $g(0,0)=6$，求 $f(0,0)$ 的值。

Let $f(x,y)$ and $g(x,y)$ be polynomials in two variables. Suppose $g(0,0)=6$, and that $(x^2+2xy+3y^2)f(x,y)=(4x^2+5xy+6y^2)g(x,y)$ for any real numbers x and y satisfying $x^2y+6y=xy^3+5x^2+2x$. Find the value of $f(0,0)$.

20. 設 a、b 為實數。已知 $2a+3$、$a+b$、$2b+3$、$a+1$、$b+1$ 和 0 這六個數中最小的兩個相等，且 $a+3$、$b+4$ 和 0 這三個數中最小的兩個也相等。求 $a+b$ 的最大可能值。

Let a, b be real numbers. It is known that the two smallest numbers among the six numbers $2a+3$, $a+b$, $2b+3$, $a+1$, $b+1$ and 0 are equal, and the two smallest numbers among the three numbers $a+3$, $b+4$ and 0 are also equal. Find the greatest possible value of $a+b$.

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