第十五屆培正數學邀請賽（2016 年）

15th Pui Ching Invitational Mathematics Competition (2016)

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

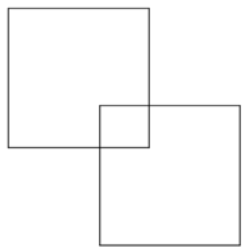
Questions 1 to 4 each carries 3 marks.

1. 在一個遊戲中，10 個紅球和 10 個藍球被放進一個袋子中，然後參加者每次抽出一個球，直至抽出所有球為止。從第二個球起，如果抽出的球的顏色和前一個抽出的球相同，便可得 1 分。若某位參加者抽出的第一個球和最後一個球都是紅色的，求他的得分的最大可能值。

   In a game, 10 red balls and 10 blue balls are put into a bag. The player then draws one ball each time until all balls are drawn. Starting from the second ball, 1 point will be awarded if the ball drawn has the same colour as the previous one. If a player draws a red ball in the first as well as the last draw, what is his maximum possible score?

2. 給定平面上兩個邊長為 16 的正方形，若兩個正方形的重疊部分是一個 4×7 的長方形，求兩個正方形中心的距離。

   Given two squares of side length 16 on the plane, if the overlapping region of the two squares is a rectangle of size 4×7, find the distance between the centres of the two squares.



3. 老師在黑板上寫上整數 2016，之後一群學生逐一進行以下操作：擦掉黑板上某個大於 1 的整數，然後寫上兩個整數，它們之差不大於 1 且它們之和等於被擦掉的整數。最後，黑板上所有的整數皆為 1。過程中總共擦掉多少個整數？

   The teacher wrote the integer 2016 on the blackboard. A group of students then performed the following operations one by one: erase an integer greater than 1 and replace it with two integers whose difference does not exceed 1 and whose sum equals the integer erased. In the end all integers on the blackboard were 1. How many integers have been erased during the process?

4. 設 n 為正整數。若 $n+18$ 、 $2n+10$ 、 $39-2n$ 、 $45-3n$ 中最大的一項是 M，求 M 的最小可能值。

Let n be a positive integer. If the largest term among $n+18$, $2n+10$, $39-2n$, $45-3n$ is M, find the smallest possible value of M.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. ABC 是等邊三角形，其中心為原點。若 A、B、C 的座標分別是 $(1,0)$ 、 (p,q) 和 (r,s) ，求 $p+q+r+s$ 的值。

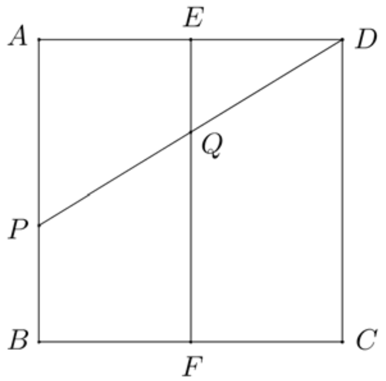
ABC is an equilateral triangle centred at the origin. If the coordinates of A, B and C are $(1,0)$, (p,q) and (r,s) respectively, find the value of $p+q+r+s$.

6. 三人進行遊戲如下：他們把九條長度分別為 8、9、⋯、16 的木棒隨機分配，使得每人有三條木棒，然後各自嘗試以所得的木棒組成三角形，並以所組成三角形周長最大者為勝。（若沒有人能組成三角形，則無人勝出。若出現和局，則以抽籤定勝負。）勝出者所組成的三角形的周長的最小可能值是多少？

Three people play a game as follows. They randomly distribute nine sticks with lengths 8, 9, …, 16 respectively among themselves so that each has three sticks. Each player then tries to form a triangle with the three sticks obtained and the one whose triangle has the greatest perimeter wins. (If nobody can form a triangle, there is no winner. If there is a tie, the winner is determined by drawing lots.) What is the smallest possible perimeter of the triangle formed by the winner?

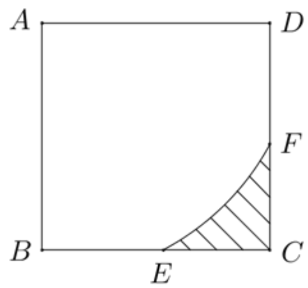
7. 圖中，$ABCD$ 是邊長為 8 的正方形，點 E 和 F 分別在邊 AD 和 BC 上，使得 EF 和 AB 平行。P 是邊 AB 上的點，使得 $AP=6$。Q 是 PD 和 EF 的交點。若四邊形 $APQE$ 和 $CDQF$ 的面積相同，求 EQ 的長度。

In the figure, $ABCD$ is a square of side length 8. Points E and F lie on sides AD and BC respectively such that EF and AB are parallel. P is a point on side AB with $AP=6$. Q is the intersection of PD and EF. If quadrilaterals $APQE$ and $CDQF$ have the same area, find the length of EQ.



8. 設 *ABCD* 是邊長為 3 的正方形。如圖所示，以 *A* 為中心作半徑為 $2\sqrt{3}$ 的弧，分別交 *BC* 和 *CD* 於 *E* 和 *F*。求圖中陰影部份的面積。

Let *ABCD* be a square of side length 3. As shown in the figure, an arc with centre *A* and radius $2\sqrt{3}$ is constructed, cutting *BC* and *CD* at *E* and *F* respectively. Find the area of the shaded region in the figure.



第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 班上有四名學生，編號為 1 至 4，每名編號為 *n* 的學生寫下一個不超過 $n+1$ 的正整數。若四名學生所寫下的整數兩兩不同，問這四個數有多少組不同的可能性？

In a class there are four students, numbered 1 to 4. Each student whose number is *n* wrote a positive integer which does not exceed $n+1$. If it turns out that the integers written by the four students are pairwise distinct, how many different possibilities are there for the four numbers?

10. 現於平面上給定兩點。有多少種不同方法可以在平面上另選 2014 個點，使得它們連同給定的兩點組成一個正 2016 邊形？

There are two given points on the plane. How many different ways are there to choose 2014 other points on the plane so that they together with the given points form a regular polygon with 2016 sides?

11. 某袋子中裝有 99 個球，分別編號為 01、02、⋯、99。現要從中選出五個球並把它們的編號作為「幸運號碼」，且須滿足以下條件：把每個「幸運號碼」的兩位數字左右倒轉後，所得的仍是一個「幸運號碼」。那麼，選球的方法共有多少種？

There are 99 balls in a bag, numbered 01, 02, …, 99 respectively. Now 5 balls are to be chosen and their numbers be designated as ‘lucky numbers’, with the requirement that by swapping the two digits of each ‘lucky number’, the result must still be a ‘lucky number’. How many ways are there to choose the balls?

12. 有多少對正整數 m 和 n 滿足 $mn=1800$，且 m 是 n 的因數？

How many pairs of positive integers m and n are there such that $mn=1800$ and m is a factor of n?

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 現有兩個直角三角形 ΔABC 和 ΔXYZ，已知 ΔXYZ 的兩條直角邊 XY 和 YZ 分別是 ΔABC 的兩條直角邊 AB 和 BC 的 25 倍和 45 倍，且 ΔXYZ 斜邊 XZ 的長度是 ΔABC 斜邊 AC 的 39 倍，求 $\frac{XZ}{AB}$ 的值。

There are two right-angled triangles ΔABC and ΔXYZ. Given that the lengths of the two legs XY and YZ of ΔXYZ are 25 times and 45 times those of the legs AB and BC of ΔABC respectively, and the length of the hypotenuse XZ of ΔXYZ is 39 times that of the hypotenuse AC of ΔABC. Find the value of $\frac{XZ}{AB}$.

14. 對正整數 n，設 $f(n)$ 表示 $\sqrt[3]{6n+\sqrt[3]{6n}}$ 的整數部份，例如：當 $n=3$ 時，$\sqrt[3]{6n+\sqrt[3]{6n}}=\sqrt[3]{18+\sqrt[3]{18}}\approx 2.7$，故此 $f(3)=2$。有多少個正整數 n 滿足 $f(n)=2016$？

For positive integer n, let $f(n)$ denote the integer part of $\sqrt[3]{6n+\sqrt[3]{6n}}$. For example, when $n=3$, we have $\sqrt[3]{6n+\sqrt[3]{6n}}=\sqrt[3]{18+\sqrt[3]{18}}\approx 2.7$ and so $f(3)=2$. How many positive integers n are there such that $f(n)=2016$?

15. 求 $\frac{2}{3}+\frac{1}{3}\times\frac{2}{4}+\frac{1}{3}\times\frac{2}{4}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\frac{2}{6}+\cdots+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\cdots\times\frac{97}{99}\times\frac{2}{100}$ 的值。

Evaluate $\frac{2}{3}+\frac{1}{3}\times\frac{2}{4}+\frac{1}{3}\times\frac{2}{4}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\frac{2}{6}+\cdots+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\cdots\times\frac{97}{99}\times\frac{2}{100}$.

16. 現有 100 張咭片，分別寫上 00、01、…、99。若隨機抽出兩張，則在兩張咭片上分別出現同一個數字的概率是多少？

There are 100 cards, with 00, 01, …, 99 written on them respectively. If two cards are chosen at random, what is the probability that there is a same digit appearing on both cards?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 老師心中想著兩個不超過 5 的正整數 m 和 n（可能相同），並分別告訴陳同學 m 和 n 之和、告訴李同學 m 和 n 中較小的一個、告訴張同學 m 和 n 中較大的一個及告訴王同學 m 和 n 之差。以下是他們之間的對話：

    老師：「你們知道這兩個數是甚麼嗎？告訴我兩個數值就可以，不必指出哪個是 m 哪個是 n。」

    陳同學：「我不知道。」

    李同學：「我不知道。」

    張同學：「我不知道。」

    王同學：「我不知道。」

    陳同學：「我還是不知道。」

    李同學：「我亦然。」

    假設學生們都是聰明的，能按照其他學生的說話作出正確分析，求 m^2+n^2 的所有可能值之和。

    The teacher had in mind two positive integers m and n (possibly equal) not greater than 5. He then told Ann the sum of m and n, told Ben the smaller integer among m and n, told Cat the larger integer among m and n, and told Dan the difference between m and n. Here is their conversation:

    ‘Do you know what the two numbers are?’ the teacher asked. ‘Just tell me the two numbers. You don’t have to point out which one is m and which one is n.’

    ‘I don’t know,’ said Ann.

    ‘I don’t know,’ said Ben.

    ‘I don’t know,’ said Cat.

    ‘I don’t know,’ said Dan.

    ‘I still don’t know,’ said Ann.

    ‘Neither do I,’ said Ben.

    Suppose the students are clever enough to analyse correctly what the previous students have said. Find the sum of all possible values of m^2+n^2.

18. 果汁每瓶售價 1 元。小華有 2016 元，他計劃每天喝一瓶果汁。每喝完一瓶果汁後，空瓶、瓶蓋和瓶身上的印花都可以儲起來，每 3 個空瓶、4 個瓶蓋或 5 枚印花可免費換領一瓶果汁。小華的計劃最多可以持續多少天？

Each bottle of juice is sold at $1. John has $2016 and plans to drink one bottle of juice every day. After finishing a bottle of juice, the empty bottle, the bottle cap and the trading stamp on the bottle can be saved for further redemption, such that every 3 empty bottles or 4 bottle caps or 5 trading stamps can be exchanged for a free bottle of juice. What is the maximum number of days for which John's plan can last?

19. 設 a、b 為實數。已知 $2a+3$、$a+b$、$2b+3$、$a+1$、$b+1$ 和 0 這六個數中最小的兩個相等，且 $a+3$、$b+4$ 和 0 這三個數中最小的兩個也相等。求 $a+b$ 的最大可能值。

Let a, b be real numbers. It is known that the two smallest numbers among the six numbers $2a+3$, $a+b$, $2b+3$, $a+1$, $b+1$ and 0 are equal, and the two smallest numbers among the three numbers $a+3$, $b+4$ and 0 are also equal. Find the greatest possible value of $a+b$.

20. 求最小的正整數 n，使得 n^n 的最後三位數字是 979。

Find the smallest positive integer n such that the last three digits of n^n are 979.

全卷完

END OF PAPER