第十五屆培正數學邀請賽（**2016** 年）

15th Pui Ching Invitational Mathematics Competition (2016)

決賽（中二組）

Final Event (Secondary 2)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 設 n 為正整數。若 $n+18$、$2n+10$、$39-2n$、$45-3n$ 中最大的一項是 M，求 M 的最小可能值。

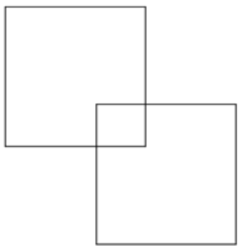
   Let n be a positive integer. If the largest term among $n+18$, $2n+10$, $39-2n$, $45-3n$ is M, find the smallest possible value of M.

2. 在一個遊戲中，10 個紅球和 10 個藍球被放進一個袋子中，然後參加者每次抽出一個球，直至抽出所有球為止。從第二個球起，如果抽出的球的顏色和前一個抽出的球相同，便可得 1 分。若某位參加者抽出的第一個球和最後一個球都是紅色的，求他的得分的最小可能值。

   In a game, 10 red balls and 10 blue balls are put into a bag. The player then draws one ball each time until all balls are drawn. Starting from the second ball, 1 point will be awarded if the ball drawn has the same colour as the previous one. If a player draws a red ball in the first as well as the last draw, what is his minimum possible score?

3. 給定平面上兩個邊長為 16 的正方形，若兩個正方形的重疊部分是一個 4×7 的長方形，求兩個正方形中心的距離。

   Given two squares of side length 16 on the plane, if the overlapping region of the two squares is a rectangle of size 4×7, find the distance between the centres of the two squares.



4. 現有 2016 枝相同長度的火柴。它們可用以組成 672 個總面積為 A 的等邊三角形，亦可用以組成 336 個總面積為 B 的正六邊形。求 $\frac{A}{B}$ 的值。

   There are 2016 matches of equal length. They can form 672 equilateral triangles with total area A, or 336 regular hexagons with total area B. Find the value of $\frac{A}{B}$.

第 5 至第 8 題，每題 4 分。

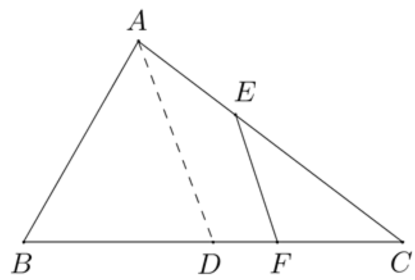
Questions 5 to 8 each carries 4 marks.

5. ABC 是等邊三角形，其中心為原點。若 A、B、C 的座標分別是 $(1,0)$ 、(p,q) 和 (r,s) ，求 $p+q+r+s$ 的值。

   ABC is an equilateral triangle centred at the origin. If the coordinates of A, B and C are $(1,0)$, (p,q) and (r,s) respectively, find the value of $p+q+r+s$.

6. 圖中，D 是 BC 的中點，E 和 F 分別位於線段 AC 和 DC 上，使得 EF 與 AD 平行。若 $ABFE$ 和 ΔCEF 的面積比是 $23:9$ ，求 $\frac{BF}{FC}$ 的值。

   In the figure, D is the mid-point of BC. E and F lie on segments AC and DC respectively, such that EF and AD are parallel. If the ratio of the area of $ABFE$ to that of ΔCEF is $23:9$, find the value of $\frac{BF}{FC}$.



7. 小美寫下了一個兩位數 n，之後小麗用以下方法構作一個數列：分別以 n 的十位和個位作為數列的第一項和第二項，然後從第三項起，每項均等於前兩項之和（例如：若 $n=37$ ，則數列為 3, 7, 10, 17, 27, …）。若數列的第 2015 項和第 2016 項的個位數字分別是 4 和 8，求 n 的值。

   Mimi writes down a two-digit number n, and then Lily constructs a sequence as follows. The tens digit and unit digit of n are used as the first term and the second term of the sequence respectively. Starting from the third term, each term is equal to the sum of the previous two terms. (For example, if $n=37$, then the sequence will be 3, 7, 10, 17, 27, …) If the unit digits of the 2015th term and the 2016th term of the sequence are 4 and 8 respectively, find the value of n.

8. 現把 2016 個單位正方體黏合成一個長方體，然後把長方體的表面塗上紅色。若剛好有兩面被塗上紅色的單位正方體共有 n 個，求 n 的最大可能值。

   2016 unit cubes are stuck together to form a cuboid. The surface of the cuboid is then painted red. If there are n unit cubes with exactly two faces painted red, find the greatest possible value of n.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 班上有四名學生，編號為 1 至 4，每名編號為 n 的學生寫下一個不超過 $n+1$ 的正整數。若四名學生所寫下的整數兩兩不同，問這四個數有多少組不同的可能性？

   In a class there are four students, numbered 1 to 4. Each student whose number is n wrote a positive integer which does not exceed $n+1$. If it turns out that the integers written by the four students are pairwise distinct, how many different possibilities are there for the four numbers?

10. 三人進行遊戲如下：他們把九條長度分別為 1、2、⋯、9 的木棒隨機分配，使得每人有三條木棒，然後各自嘗試以所得的木棒組成三角形，並以所組成三角形周長最大者為勝。（若沒有人能組成三角形，則無人勝出。若出現和局，則以抽籤定勝負。）勝出者所組成的三角形的周長的最小可能值是多少？

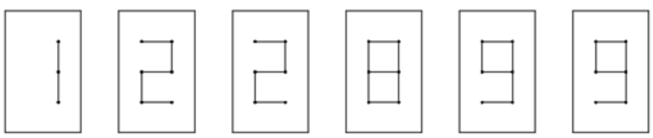
    Three people play a game as follows. They randomly distribute nine sticks with lengths 1, 2, …, 9 respectively among themselves so that each has three sticks. Each player then tries to form a triangle with the three sticks obtained and the one whose triangle has the greatest perimeter wins. (If nobody can form a triangle, there is no winner. If there is a tie, the winner is determined by drawing lots.) What is the smallest possible perimeter of the triangle formed by the winner?

11. 設 n 為正整數。若把其正因數從小至大排列，首四個依次是 a、b、c、d。若 $n=(bd+a)(c^2-a)$，求 n 的值。

    Let n be a positive integer. When its positive factors are listed in ascending order, the first four terms are a, b, c and d in order. If $n=(bd+a)(c^2-a)$, find the value of n.

12. 利用以下 6 張印有電子數字的透明咭片共可組成多少個六位數？（註：透明咭片可倒轉或反轉使用。）

    How many six-digit numbers can be formed using the following 6 transparent cards with digital numbers? (Note: The transparent cards may be put upside down or turned over.)



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 對正整數 n，設 $f(n)$ 表示 $\sqrt{2n+\frac{1}{4}}$ 四捨五入至最接近整數後所得的值，例如：當 $n=4$ 時，$\sqrt{2n+\frac{1}{4}}=\sqrt{8.25}\approx 2.9$，故此 $f(4)=3$。求 $f(1)+f(2)+\cdots+f(30)$ 的值。

    For positive integer n, let $f(n)$ denote the result of rounding off $\sqrt{2n+\frac{1}{4}}$ to the nearest integer. For example, when $n=4$, we have $\sqrt{2n+\frac{1}{4}}=\sqrt{8.25}\approx 2.9$ and so $f(4)=3$. Find the value of $f(1)+f(2)+\cdots+f(30)$.

14. 現有兩個直角三角形 ΔABC 和 ΔXYZ，已知 ΔXYZ 的兩條直角邊 XY 和 YZ 分別是 ΔABC 的兩條直角邊 AB 和 BC 的 25 倍和 45 倍，且 ΔXYZ 斜邊 XZ 的長度是 ΔABC 斜邊 AC 的 39 倍，求 $\frac{XZ}{AB}$ 的值。

    There are two right-angled triangles ΔABC and ΔXYZ. Given that the lengths of the two legs XY and YZ of ΔXYZ are 25 times and 45 times those of the legs AB and BC of ΔABC respectively, and the length of the hypotenuse XZ of ΔXYZ is 39 times that of the hypotenuse AC of ΔABC. Find the value of $\frac{XZ}{AB}$.

15. 有多少個只由 1、2、3、4、5 或 6 組成的七位數的末兩位是 66，且其餘連續數位均不為 66？

    How many seven-digit numbers composed only of 1, 2, 3, 4, 5 or 6 are there, which end in 66 and in which all other consecutive digits are not 66?

16. 設 a、b、c 為非零實數，其中 $3a+2b-c=-4$。若實數 m 同時是三條方程 $ax^3+x^2-3x+1=0$、$4x^3+bx^2+2x-1=0$ 和 $cx^3+3x^2-7x+2=0$ 的解，求 m 的值。

    Let a, b, c be nonzero real numbers such that $3a+2b-c=-4$. If the real number m is a solution to each of the three equations $ax^3+x^2-3x+1=0$, $4x^3+bx^2+2x-1=0$ and $cx^3+3x^2-7x+2=0$, find the value of m.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 老師心中想著兩個不超過 5 的正整數 m 和 n（可能相同），並分別告訴陳同學 m 和 n 之和、告訴李同學 m 和 n 中較小的一個、告訴張同學 m 和 n 中較大的一個及告訴王同學 m 和 n 之差。以下是他們之間的對話：

    老師：「你們知道這兩個數是甚麼嗎？告訴我兩個數值就可以，不必指出哪個是 m 哪個是 n。」

    陳同學：「我不知道。」

    李同學：「我不知道。」

    張同學：「我不知道。」

    王同學：「我不知道。」

    陳同學：「我還是不知道。」

    李同學：「我亦然。」

    假設學生們都是聰明的，能按照其他學生的說話作出正確分析，求 m^2+n^2 的所有可能值之和。

    The teacher had in mind two positive integers m and n (possibly equal) not greater than 5. He then told Ann the sum of m and n, told Ben the smaller integer among m and n, told Cat the larger integer among m and n, and told Dan the difference between m and n. Here is their conversation:

    'Do you know what the two numbers are?' the teacher asked. 'Just tell me the two numbers. You don't have to point out which one is m and which one is n.'

    'I don't know,' said Ann.

    'I don't know,' said Ben.

    'I don't know,' said Cat.

    'I don't know,' said Dan.

    'I still don't know,' said Ann.

    'Neither do I,' said Ben.

    Suppose the students are clever enough to analyse correctly what the previous students have said. Find the sum of all possible values of m^2+n^2.

18. 求最小的正整數 n，使得 n^n 的最後三位數字是 979。

Find the smallest positive integer n such that the last three digits of n^n are 979.

19. 果汁每瓶售價 1 元。小華有 2016 元，他計劃每天喝一瓶果汁。每喝完一瓶果汁後，空瓶、瓶蓋和瓶身上的印花都可以儲起來，每 3 個空瓶、4 個瓶蓋或 5 枚印花可免費換領一瓶果汁。小華的計劃最多可以持續多少天？

Each bottle of juice is sold at $1. John has $2016 and plans to drink one bottle of juice every day. After finishing a bottle of juice, the empty bottle, the bottle cap and the trading stamp on the bottle can be saved for further redemption, such that every 3 empty bottles or 4 bottle caps or 5 trading stamps can be exchanged for a free bottle of juice. What is the maximum number of days for which John's plan can last?

20. 某城市使用一套特別的系統表示整數，當中有 5 種不同的數碼 A、B、C、D、E，其數值分別對應於我們平常使用的十進制中的 1、6、36、216、1296。該系統中的 n 位數形如 $\overline{\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n}$，其中每個 X_i 都是 A、B、C、D、E 之一，而它等於十進制中的 $\mathrm{X}_1\pm\mathrm{X}_2\pm\cdots\pm\mathrm{X}_n$，當中若 $\mathrm{X}_i>\mathrm{X}_{i+1}$，則 X_i 和 X_{i+1} 之間取負號，否則取正號。例如，該系統中的四位數 $\overline{\mathrm{ACCB}}$ 等於十進制中的 $1+36+36-6=67$。在這個系統中的三位數共可表示多少個不同的值？

A city uses a special system to represent integers. In the system, there are 5 different numerals A, B, C, D, E, corresponding to the values 1, 6, 36, 216, 1296 respectively in our usual decimal system. An n-digit number in this system is of the form $\overline{\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n}$, where each X_i is one of A, B, C, D, E, and it is equivalent to the number $\mathrm{X}_1\pm\mathrm{X}_2\pm\cdots\pm\mathrm{X}_n$ in the decimal system, where the sign between each X_i and X_{i+1} is negative if $\mathrm{X}_i>\mathrm{X}_{i+1}$, and is positive otherwise. For example, the four-digit number $\overline{\mathrm{ACCB}}$ in this system is equal to $1+36+36-6=67$ in the decimal system. How many different values can be represented by the three-digit numbers in this system?

全卷完

END OF PAPER