**第十五屆培正數學邀請賽（2016 年）**

**15th Pui Ching Invitational Mathematics Competition (2016)**

**決賽（中一組）**

**Final Event (Secondary 1)**

**時限：2 小時**

**Time allowed: 2 hours**

**參賽者須知：**

**Instructions to Contestants:**

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

**第 1 至第 4 題，每題 3 分。**

**Questions 1 to 4 each carries 3 marks.**

1. 在一個遊戲中，10 個紅球和 10 個藍球被放進一個袋子中，然後參加者每次抽出一個球，直至抽出所有球為止。從第二個球起，如果抽出的球的顏色和前一個抽出的球相同，便可得 1 分。若某位參加者抽出的第一個球和最後一個球都是紅色的，求他的得分的最大可能值。

   In a game, 10 red balls and 10 blue balls are put into a bag. The player then draws one ball each time until all balls are drawn. Starting from the second ball, 1 point will be awarded if the ball drawn has the same colour as the previous one. If a player draws a red ball in the first as well as the last draw, what is his maximum possible score?

2. 嘉雯在解方程 $-1234x+5678=-9012x+3456$ 時，誤把式中的兩個「$-$」號看成「$+$」號，又誤把兩個「$+$」號看成「$-$」號。那麼，嘉雯所得的答案與正確答案之差是多少？

   When solving the equation $-1234x+5678=-9012x+3456$, Carmen mistakenly regarded the two '–' signs in the equation as '+' and the two '+' signs as '–'. What is the difference between the answer Carmen obtained and the correct answer?

3. 老師在黑板上寫上整數 2016，之後一群學生逐一進行以下操作：擦掉黑板上某個大於 1 的整數，然後寫上兩個整數，它們之差不大於 1 且它們之和等於被擦掉的整數。最後，黑板上所有的整數皆為 1。過程中總共擦掉多少個整數？

   The teacher wrote the integer 2016 on the blackboard. A group of students then performed the following operations one by one: erase an integer greater than 1 and replace it with two integers whose difference does not exceed 1 and whose sum equals the integer erased. In the end all integers on the blackboard were 1. How many integers have been erased during the process?

4. 設 $n=6^{6^6}$。若 $n^{36}=6^{6^x}$，求 $x$ 的值。

   Let $n=6^{6^6}$. If $n^{36}=6^{6^x}$, find the value of $x$.

**第 5 至第 8 題，每題 4 分。**

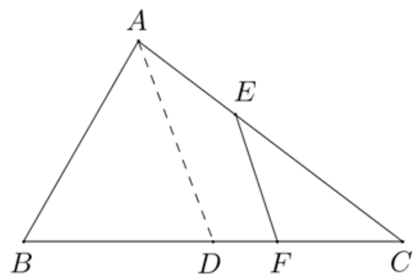
**Questions 5 to 8 each carries 4 marks.**

5. 考慮數列 2, 0, 2, 2, 4, …，其中由第 3 項開始，每項均是前兩項之和。若第 2015 項和第 2016 項的個位數分別是 $A$ 和 $B$，求兩位數 $\overline{AB}$ 的值。

   Consider the sequence 2, 0, 2, 2, 4, …, where starting from the 3rd term, each term is the sum of the two previous terms. If the unit digits of the 2015th and the 2016th terms are $A$ and $B$ respectively, find the value of the two-digit number $\overline{AB}$.

6. 圖中，$D$ 是 $BC$ 的中點，$E$ 和 $F$ 分別位於線段 $AC$ 和 $DC$ 上，使得 $EF$ 與 $AD$ 平行。若 $ABFE$ 和 $\Delta CEF$ 的面積比是 $23:9$，求 $\frac{BF}{FC}$ 的值。

   In the figure, $D$ is the mid-point of $BC$. $E$ and $F$ lie on segments $AC$ and $DC$ respectively, such that $EF$ and $AD$ are parallel. If the ratio of the area of $ABFE$ to that of $\Delta CEF$ is $23:9$, find the value of $\frac{BF}{FC}$.




7. 現把 2016 個單位正方體黏合成一個長方體，然後把長方體的表面塗上紅色。若剛好有兩面被塗上紅色的單位正方體共有 $n$ 個，求 $n$ 的最大可能值。

   2016 unit cubes are stuck together to form a cuboid. The surface of the cuboid is then painted red. If there are $n$ unit cubes with exactly two faces painted red, find the greatest possible value of $n$.

8. 在座標平面上，某個正 2016 邊形以原點為中心。若其中一個頂點的座標是 $(1,0)$，求另外 2015 個頂點的所有 $x$ 座標和 $y$ 座標之和。

   On the coordinate plane, a regular 2016-sided polygon is centred at the origin. If one of the vertices has coordinates $(1,0)$, find the sum of the $x$-coordinates and the $y$-coordinates of all other 2015 vertices.

**第 9 至第 12 題，每題 5 分。**

**Questions 9 to 12 each carries 5 marks.**

9. 有多少對正整數 $m$ 和 $n$ 滿足 $mn=1800$ ，且 $m$ 是 $n$ 的因數？

   How many pairs of positive integers $m$ and $n$ are there such that $mn=1800$ and $m$ is a factor of $n$?

10. 現於平面上給定兩點。有多少種不同方法可以在平面上另選 2014 個點，使得它們連同給定的兩點組成一個正 2016 邊形？

    There are two given points on the plane. How many different ways are there to choose 2014 other points on the plane so that they together with the given points form a regular polygon with 2016 sides?

11. 三人進行遊戲如下：他們把九條長度分別為 1、2、⋯、9 的木棒隨機分配，使得每人有三條木棒，然後各自嘗試以所得的木棒組成三角形，並以所組成三角形周長最大者為勝。（若沒有人能組成三角形，則無人勝出。若出現和局，則以抽籤定勝負。）勝出者所組成的三角形的周長的最小可能值是多少？

    Three people play a game as follows. They randomly distribute nine sticks with lengths 1, 2, …, 9 respectively among themselves so that each has three sticks. Each player then tries to form a triangle with the three sticks obtained and the one whose triangle has the greatest perimeter wins. (If nobody can form a triangle, there is no winner. If there is a tie, the winner is determined by drawing lots.) What is the smallest possible perimeter of the triangle formed by the winner?

12. 設 $n$ 為正整數。若把其正因數從小至大排列，首四個依次是 $a$、$b$、$c$、$d$。若 $n=(bd+a)(c^2-a)$ ，求 $n$ 的值。

    Let $n$ be a positive integer. When its positive factors are listed in ascending order, the first four terms are $a$, $b$, $c$ and $d$ in order. If $n=(bd+a)(c^2-a)$, find the value of $n$.

**第 13 至第 16 題，每題 6 分。**

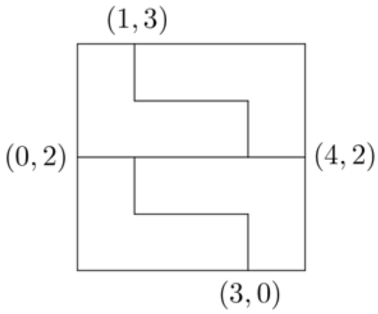
**Questions 13 to 16 each carries 6 marks.**

13. 有多少個只由 1、2、3、4、5 或 6 組成的六位數的末兩位是 65，且其餘連續數位均不為 65？

    How many six-digit numbers composed only of 1, 2, 3, 4, 5 or 6 are there, which end in 65 and in which all other consecutive digits are not 65?

14. 給定座標平面上以 $(0,0)$ 、$(4,0)$ 、$(4,4)$ 和 $(0,4)$ 為頂點的正方形，現要加上一些線段以把正方形分割成 4 個相同的連通區域（即每個區域均可通過平移和旋轉得到另一個區域），其中每條線段必須與正方形的一邊平行，且其端點的座標必須為整數（圖中給出其中一種分割正方形的方法）。那麼有多少種方法加上線段？

    A square on the coordinate plane with vertices $(0,0)$, $(4,0)$, $(4,4)$ and $(0,4)$ is given. Now line segments are to be drawn to divide the square into 4 identical connected regions (i.e. each region can be transformed to another by translation and rotation), with the restriction that each segment must be parallel to a side of the square and its endpoints must have integer coordinates. (The figure shows one possible way to divide the square.) How many different ways are there to draw the line segments?




15. 求 $\frac{2}{3}+\frac{1}{3}\times\frac{2}{4}+\frac{1}{3}\times\frac{2}{4}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\frac{2}{6}+\cdots+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\cdots\times\frac{97}{99}\times\frac{2}{100}$ 的值。

    Evaluate $\frac{2}{3}+\frac{1}{3}\times\frac{2}{4}+\frac{1}{3}\times\frac{2}{4}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\frac{2}{6}+\cdots+\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\cdots\times\frac{97}{99}\times\frac{2}{100}$.

16. 設 $a$、$b$、$c$ 為非零實數，其中 $3a+2b-c=-4$ 。若實數 $m$ 同時是三條方程 $ax^3+x^2-3x+1=0$ 、$4x^3+bx^2+2x-1=0$ 和 $cx^3+3x^2-7x+2=0$ 的解，求 $m$ 的值。

    Let $a$, $b$, $c$ be nonzero real numbers such that $3a+2b-c=-4$. If the real number $m$ is a solution to each of the three equations $ax^3+x^2-3x+1=0$, $4x^3+bx^2+2x-1=0$ and $cx^3+3x^2-7x+2=0$, find the value of $m$.

**第 17 至第 20 題，每題 7 分。**

**Questions 17 to 20 each carries 7 marks.**

17. 老師心中想著兩個不超過 5 的正整數 $m$ 和 $n$（可能相同），並分別告訴陳同學 $m$ 和 $n$ 之和、告訴李同學 $m$ 和 $n$ 中較小的一個、告訴張同學 $m$ 和 $n$ 中較大的一個及告訴王同學 $m$ 和 $n$ 之差。以下是他們之間的對話：

    老師：「你們知道這兩個數是甚麼嗎？告訴我兩個數值就可以，不必指出哪個是 $m$ 哪個是 $n$。」

    陳同學：「我不知道。」

    李同學：「我不知道。」

    張同學：「我不知道。」

    王同學：「我不知道。」

    陳同學：「我還是不知道。」

    李同學：「我亦然。」

    假設學生們都是聰明的，能按照其他學生的說話作出正確分析，求 $m^2+n^2$ 的所有可能值之和。

    The teacher had in mind two positive integers $m$ and $n$ (possibly equal) not greater than 5. He then told Ann the sum of $m$ and $n$, told Ben the smaller integer among $m$ and $n$, told Cat the larger integer among $m$ and $n$, and told Dan the difference between $m$ and $n$. Here is their conversation:

    ‘Do you know what the two numbers are?’ the teacher asked. ‘Just tell me the two numbers. You don’t have to point out which one is $m$ and which one is $n$.’

    ‘I don’t know,’ said Ann.

    ‘I don’t know,’ said Ben.

    ‘I don’t know,’ said Cat.

    ‘I don’t know,’ said Dan.

    ‘I still don’t know,’ said Ann.

    ‘Neither do I,’ said Ben.

    Suppose the students are clever enough to analyse correctly what the previous students have said. Find the sum of all possible values of $m^2+n^2$.

18. 在一個 $5\times5$ 方格表的每格內各有 10 枚金幣。在每一步中，我們可以選擇同一行或同一列 4 個相連且各有最少一枚金幣的方格，然後從這 4 個方格內各取走一枚金幣。按此規則，經過若干步後最多可取走多少枚金幣？

In each cell of a $5\times5$ grid, there are 10 coins. In each step, one can choose 4 consecutive cells in a row or in a column, each of which contains at least one coin, and then take away a coin from each of these 4 cells. At most how many coins can be taken away after some steps according to such regulation?

19. 對正整數 $n$，設 $f(n)$ 表示 $\sqrt[3]{6n+\sqrt[3]{6n}}$ 的整數部份，例如：當 $n=3$ 時，$\sqrt[3]{6n+\sqrt[3]{6n}}=\sqrt[3]{18+\sqrt[3]{18}}\approx 2.7$，故此 $f(3)=2$。求 $f(1)+f(2)+\cdots+f(2016)$ 的值。

For positive integer $n$, let $f(n)$ denote the integer part of $\sqrt[3]{6n+\sqrt[3]{6n}}$. For example, when $n=3$, we have $\sqrt[3]{6n+\sqrt[3]{6n}}=\sqrt[3]{18+\sqrt[3]{18}}\approx 2.7$ and so $f(3)=2$. Find the value of $f(1)+f(2)+\cdots+f(2016)$.

20. 某城市使用一套特別的系統表示整數，當中有 5 種不同的數碼 A、B、C、D、E，其數值分別對應於我們平常使用的十進制中的 1、6、36、216、1296。該系統中的 $n$ 位數形如 $\overline{\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n}$，其中每個 $\mathrm{X}_i$ 都是 A、B、C、D、E 之一，而它等於十進制中的 $\mathrm{X}_1\pm\mathrm{X}_2\pm\cdots\pm\mathrm{X}_n$，當中若 $\mathrm{X}_i>\mathrm{X}_{i+1}$，則 $\mathrm{X}_i$ 和 $\mathrm{X}_{i+1}$ 之間取負號，否則取正號。例如，該系統中的四位數 $\overline{\mathrm{ACCB}}$ 等於十進制中的 $1+36+36-6=67$。在這個系統中的三位數共可表示多少個不同的值？

A city uses a special system to represent integers. In the system, there are 5 different numerals A, B, C, D, E, corresponding to the values 1, 6, 36, 216, 1296 respectively in our usual decimal system. An $n$-digit number in this system is of the form $\overline{\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n}$, where each $\mathrm{X}_i$ is one of A, B, C, D, E, and it is equivalent to the number $\mathrm{X}_1\pm\mathrm{X}_2\pm\cdots\pm\mathrm{X}_n$ in the decimal system, where the sign between each $\mathrm{X}_i$ and $\mathrm{X}_{i+1}$ is negative if $\mathrm{X}_i>\mathrm{X}_{i+1}$, and is positive otherwise. For example, the four-digit number $\overline{\mathrm{ACCB}}$ in this system is equal to $1+36+36-6=67$ in the decimal system. How many different values can be represented by the three-digit numbers in this system?

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**END OF PAPER**