第十四屆培正數學邀請賽（2015 年）

14th Pui Ching Invitational Mathematics Competition (2015)

初賽（高中組）

Heat Event (Senior Secondary)

時限：1 小時 15 分

Time allowed: 1 hour 15 minutes

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 作答時，每題的答案均須以 0 至 9999 之間的整數表示。依照答題紙上的指示填寫答案，毋須呈交計算步驟。

Each answer must be given in the form of an integer between 0 and 9999. Follow the instructions on the answer sheet to enter the answers. You are not required to hand in your steps of working.

(d) 不得使用計算機。

The use of calculators is not allowed.

(e) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：每題的答案均須以 0 至 9999 之間的整數表示，如有需要應以上述範圍內最接近正確答案的整數回答。如有兩個這樣的整數與正確答案同樣接近，則以「四捨五入」的原則取較大的整數。請細閱答題紙上的指示。

Note: Each answer must be given in the form of an integer between 0 and 9999. Where necessary, the answer should be rounded off to the nearest integer in the above range. Read the instructions on the answer sheet in detail.

1. 求方程 $x^2 - 2015x + 1 = 0$ 的最大實根。 （3分）

   Find the greatest real root to the equation $x^2 - 2015x + 1 = 0$. (3 marks)

2. 有多少個小於 300 的三位平方數，其三位數字互不相同？ （3分）

   How many three-digit positive square numbers smaller than 300 are there whose three digits are pairwise distinct? (3 marks)

3. 一個等腰直角三角形的斜邊長度是 100。求它的面積。 （3分）

   The hypotenuse of an isosceles right-angled triangle is 100. Find its area. (3 marks)

4. 某國每年在四月份設 4 天公眾假期。貝詩於某年四月前往該國旅遊連續 n 天。雖然不知道哪幾天是公眾假期，但貝詩卻可以肯定於旅遊期間必定遇上最少一天公眾假期。求 n 的最小可能值。 （3分）

   In a country, there are 4 days of public holidays in April each year. Elizabeth travels to that country in April of a certain year for a continuous period of n days. Although she doesn't know the exact dates of the public holidays, Elizabeth is sure to encounter at least one public holiday during her trip. Find the smallest possible value of n. (3 marks)

5. 有多少個四位正整數的每位數字均小於 3，且該數不能被 10 整除？ （4分）

   How many four-digit positive integers are there such that each of its digits is less than 3 and that it is not divisible by 10? (4 marks)

6. 若 $f(x) = x^{10}$，求 $f^{(6)}(0.1)$ 的值。 （4分）

   If $f(x) = x^{10}$, find the value of $f^{(6)}(0.1)$. (4 marks)

7. 設 x、y 為實數。若 $x^2 + y^2 - 4x = 12$，求 x 的最大可能值。 （4分）

   Let x and y be real numbers. If $x^2 + y^2 - 4x = 12$, find the greatest possible value of x. (4 marks)

8. 某三角形三條邊的長度為 3、5 和 7。若它最大的一隻內角是 $x°$，求 x。 （4分）

   A triangle has side lengths 3, 5 and 7. If its largest interior angle is $x°$, find x. (4 marks)

9. 一個水杯的容量為 a 毫升。開始時水杯是空的，現在卓瑩將 2015 毫升水注入杯內，再將杯中一半的水倒走，然後又將 2015 毫升水注入杯內，將杯中一半的水倒走，不斷重複此動作。已知不論卓瑩重複操作多少次，注水時都不會有水從杯中溢出。求 a 的最小可能值。 （5分）

   A container has capacity a mL. Initially the container is empty. Now Betty pours 2015mL water into the container, and then pours half of the water out of the container. Next, she again pours 2015mL of water into the container, and then pours half of the water out of the container. The same action is repeated many times. It is known that no matter how many times Betty repeats this operation, overflow will not occur when water is poured into the container. Find the smallest possible value of a. (5 marks)

10. 設 n 為正整數，且 $\log(n-30) + \log(50-n) < 2$。那麼，$n$ 有多少個不同的可能值？ （5分）

    If n is a positive integer such that $\log(n-30) + \log(50-n) < 2$, how many different possible values of n are there? (5 marks)

11. 給定 $\triangle ABC$，其中 $\angle ACB = 90°$ 和 $\angle ABC = 14°$。點 O 在邊 BC 上，以 O 為中心的半圓與 AC 相切於 C、與 AB 相切於 D。若線段 AO 與半圓相交於 E，CF 為半圓的直徑，並設 $\angle EDF = x°$，求 x 的值。 （5分）

    Given $\triangle ABC$ where $\angle ACB = 90°$ and $\angle ABC = 14°$. Point O lies on side BC. The semi-circle with centre O is tangent to AC at C, and is tangent to AB at D. Suppose segment AO meets the semi-circle at E and CF is the diameter of the semi-circle. Let $\angle EDF = x°$. Find the value of x. (5 marks)
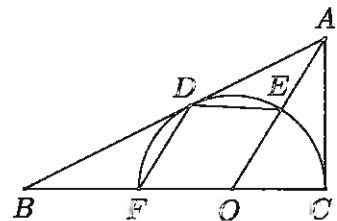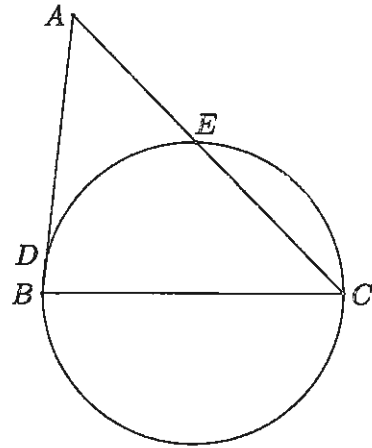
12. 已知 n 以七進制和九進制表示時均為三位數，且兩種表示方式下的三位數字恰好左右倒轉。求 n 的值。 （5分）

Suppose both the base 7 and base 9 representations of n are three-digit numbers, and the three digits in the two representations are reversed. Find the value of n. (5 marks)

13. 以 BC 為直徑的圓分別與 $\triangle ABC$ 的邊 AB 和 AC 相交於 D 和 E。若 $BC = \sqrt{901}$，$BD = 1$ 及 $AD = 16$，求 CE 的長度。 （6分）

The circle with diameter BC meets the sides AB and AC of $\triangle ABC$ at D and E respectively. If $BC = \sqrt{901}$, $BD = 1$ and $AD = 16$, find the length of CE. (6 marks)

14. 在聖誕聯歡會上，5 名小孩進行禮物交換。他們每人先將一份禮物交給老師，然後老師隨機將禮物分配給小孩。若恰好有一名小孩得到自己禮物的概率以最簡分數表示時為 $\dfrac{a}{b}$，求 $a+b$ 的值。 （6分）

In a Christmas party, 5 kids are going to exchange presents. Each of them first gives a present to the teacher, and then the teacher randomly distributes the presents back to the kids. If the probability that exactly one kid gets back his own present is $\dfrac{a}{b}$ in lowest form, find the value of $a+b$. (6 marks)

15. 現從一個八角錐體中任意選取其中三個頂點。若當中有至少兩點在同一條邊上的概率以最簡分數表示時為 $\dfrac{a}{b}$，求 $a+b$ 的值。 （6分）

We randomly choose three vertices from an octagonal pyramid. If the probability that at least two chosen vertices lie on the same edge is $\dfrac{a}{b}$ in lowest form, find the value of $a+b$. (6 marks)

16. 現將 2、3、4、5、6、7、8 分成兩組。若兩組的各數之積相差 m，求 m 的最小可能值。 （6分）

The numbers 2, 3, 4, 5, 6, 7, 8 are divided into two groups. If the difference between the products of the numbers in the two groups is m, find the smallest possible value of m. (6 marks)

17. 若 p、q、r 爲質數，且 $p^2+q^2+r^2=7875$，求 p 的最大可能值。 （7分）

If p, q, r are prime numbers such that $p^2+q^2+r^2=7875$, find the greatest possible value of p. (7 marks)

18. 設 n 爲正整數，且 n^2 是四位數。若 n^2 的千位、百位和十位都是奇數，且數字之和是 25，求 n 所有可能值之和。 （7分）

Let n be a positive integer such that n^2 is a four-digit number. If n^2 has sum of digits 25 with its thousands, hundreds and tens digits being odd, find the sum of all possible values of n. (7 marks)

19. 設 x、y、z 爲實數，其中 $(x-15)^2+(y-20)^2 \le 49$ 而 $10y=zx$。求 z 的最大可能值。 （7分）

Let x, y, z be real numbers such that $(x-15)^2+(y-20)^2 \le 49$ and $10y=zx$. Find the greatest possible value of z. (7 marks)

20. 設 $f(x)$ 爲定義在區間 $\left[0,\dfrac{\pi}{2}\right]$ 上的連續函數，其中對任意 $0<x\le\dfrac{\pi}{2}$，皆有 $f(x)=\dfrac{20x}{\sin x}$。求 $f(x)$ 的最大值和最小值之差。 （7分）

Let $f(x)$ be a continuous function defined on the interval $\left[0,\dfrac{\pi}{2}\right]$, such that $f(x)=\dfrac{20x}{\sin x}$ for any $0<x\le\dfrac{\pi}{2}$. Find the difference between the maximum and minimum values of $f(x)$. (7 marks)

全卷完

END OF PAPER