第十四屆培正數學邀請賽（2015 年）

14th Pui Ching Invitational Mathematics Competition (2015)

決賽（高中組）

Final Event (Senior Secondary)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 設 $a_1, a_2, \ldots, a_{2015}$ 為等差數列。若 $a_3 + a_{100} = 212$ 而 $a_5 + a_{2014} = 4044$，求 a_1 的值。

   Let $a_1, a_2, \ldots, a_{2015}$ be an arithmetic sequence. If $a_3 + a_{100} = 212$ and $a_5 + a_{2014} = 4044$, find the value of a_1.

2. 一組 40 人來到河岸邊，岸邊有一艘最多能載四人的小船，小船必須由一人駕駛。那麼最少需要駕駛多少次才可讓所有人過河（小船由河的一邊駛至另一邊時算作一次）？

   A group of 40 people arrive at a river bank where there is a single boat that can carry a maximum of four people. The boat has to be operated by one person. What is the minimum number of boat trips needed for all the people to cross the river? (When the boat travels from one side of the river to the other side it is counted as one trip.)

3. 某班 60 名學生排成一行，每個回合他們由左至右從 1 開始報數，報出 5 的倍數的學生需離開，然後重複進行下一回合，直至餘下的學生數目小於 5 為止。那麼，這個過程需經過多少個回合後才會停止？

   The 60 students of a class are aligned in a row. In each round, they count from 1 onwards from left to right. The students counting multiples of 5 have to leave. This process is then repeated in the next round and so on, until the number of remaining students is less than 5. How many rounds will this process last before it comes to a stop?

4. 設 n 為三位平方數，其數字之和是 6 的倍數。求 n 的最小可能值。

   Let n be a three-digit square number whose sum of digits is a multiple of 6. Find the smallest possible value of n.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 在一個數學比賽中，其中一道題目要求參賽者找出 $\frac{1007}{2015}$ 小數點後第 100000 個位。甲和乙是兩個參賽者，已知甲找出正確的答案，而乙把分數錯誤看成 $\frac{1008}{2015}$，那麼甲和乙的答案之和是甚麼？

   In a mathematical competition, one question asks contestants to find the 100000-th digit after the decimal point if $\frac{1007}{2015}$ is written as a decimal. Alice and Bob were two contestants for the competition, with Alice getting the correct answer but Bob misreading the fraction as $\frac{1008}{2015}$. What is the sum of Alice's and Bob's answers?

6. 若方程 $x^2 - 100x + k = 0$（其中 k 是整數）的兩根均為質數，求 k 所有可能值之和。

   If the two roots to the equation $x^2 - 100x + k = 0$ (where k is an integer) are both prime numbers, find the sum of all possible values of k.

7. 穿過座標平面上 $(1,1)$ 、$(1,7)$ 和 $(9,1)$ 三點的圓的半徑是多少？

   What is the radius of the circle passing through the three points $(1,1)$, $(1,7)$ and $(9,1)$ on the coordinate plane?

8. 一個等邊三角形內接於一個面積為 1 的圓形。求等邊三角形的面積。

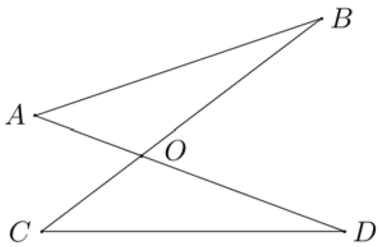
   An equilateral triangle is inscribed in a circle of area 1. Find the area of the equilateral triangle.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 圖中，AOD 和 BOC 為直線，$OA = OC = 4$，$OB = 10$，$OD = 5$，$CD = 8$。求 AB 的長度。

   In the figure, AOD and BOC are straight lines, $OA = OC = 4$, $OB = 10$, $OD = 5$ and $CD = 8$. Find the length of AB.



10. 設 α 、 β 、 γ 為方程 $x^3-20x-15=0$ 的三個根。求 $\alpha^3+\beta^3+\gamma^3$ 的值。

Let α , β , γ be the three roots to the equation $x^3-20x-15=0$. Find the value of $\alpha^3+\beta^3+\gamma^3$.

11. 求 $(3\sin x°-4\cos x°-10)(3\sin x°+4\cos x°-10)$ 的最小值。

Find the smallest possible value of $(3\sin x°-4\cos x°-10)(3\sin x°+4\cos x°-10)$.

12. 若 $\log_2(-3\sin x°)=2\log_2(\cos x°)+1$ ，其中 $0\le x\le 360$ ，求 x 的所有可能值之和。

If $\log_2(-3\sin x°)=2\log_2(\cos x°)+1$ where $0\le x\le 360$, find the sum of all possible values of x.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 一個袋子中有 12 個球，包括 3 個白球、4 個紅球和 5 個綠球。若從袋中隨機抽出 5 個球，當中最少有 3 個球同色的概率是多少？

There are 12 balls in a bag, including 3 white, 4 red and 5 green. If 5 balls are picked randomly from the bag, what is the probability that at least three of them are of the same colour?

14. 若 a、b、c、d 為正整數，使得 $ab=cd<1000$ 及 $a+b=c+d+1$，求 ab 的最大可能值。

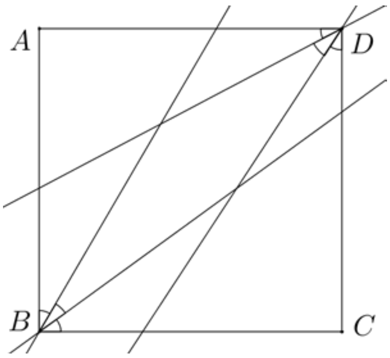
Let a, b, c, d be positive integers such that $ab=cd<1000$ and $a+b=c+d+1$. Find the greatest possible value of ab.

15. 求 $\tan^{-1}\frac{1}{2\times1^2}+\tan^{-1}\frac{1}{2\times2^2}+\tan^{-1}\frac{1}{2\times3^2}+\tan^{-1}\frac{1}{2\times4^2}+\cdots$ 的值。

Find the value of $\tan^{-1}\frac{1}{2\times1^2}+\tan^{-1}\frac{1}{2\times2^2}+\tan^{-1}\frac{1}{2\times3^2}+\tan^{-1}\frac{1}{2\times4^2}+\cdots$.

16. $ABCD$ 是邊長 1 的正方形。$\angle ABC$ 和 $\angle ADC$ 各自的兩條三等分線圍成一個四邊形。求該四邊形的面積。

$ABCD$ is a square of side length 1. The two trisectors of each of $\angle ABC$ and $\angle ADC$ form a quadrilateral. Find the area of that quadrilateral.



第 17 至第 20 題，每題 7 分。

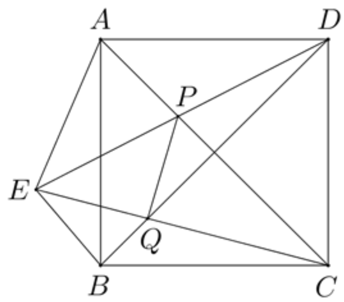
Questions 17 to 20 each carries 7 marks.

17. 在一個正方形內任意選取一點 P，並連接 P 和正方形的四個頂點，以把正方形分成四個三角形。求四個三角形的所有內角均不超過 $120°$ 的概率，答案準確至小數點後 2 位（提示：$\sqrt{3} \approx 1.73205$）。

A point P is chosen randomly in a square. Join P with the four vertices of the square so as to divide the square into four triangles. Find, correct to 2 decimal places, the probability that all interior angles of the four triangles do not exceed $120°$. (Hint: $\sqrt{3} \approx 1.73205$.)

18. 圖中，$ABCD$ 是正方形，E 是正方形外的一點，使得 $\angle AEB = 135°$、$AE = 6$ 及 $EB = 4$。若 DE 和 AC 相交於 P，CE 和 BD 相交於 Q，求 $CDPQ$ 的面積。

In the figure, $ABCD$ is a square. E is a point outside the square such that $\angle AEB = 135°$, $AE = 6$ and $EB = 4$. If DE and AC intersect at P while CE and BD intersect at Q, find the area of $CDPQ$.



19. 若 $\int_{-1}^{x} [t^3 - 6042t^2 + (2015 \times 4024 + 3)t + 4029 \times 2014]\, dt = 0$，求 x 的最大可能值。

If $\int_{-1}^{x} [t^3 - 6042t^2 + (2015 \times 4024 + 3)t + 4029 \times 2014]\, dt = 0$, find the greatest possible value of x.

20. 設 $P(x)$ 為實系數多項式，其中 $P(0)=0$ 及 $P(1)=1$，另設 $f(x)=[P'(x)]^2-20xP'(x)$。求 $\int_0^1 f(x)\,dx$ 的最小可能值。

Let $P(x)$ be a polynomial with real coefficients such that $P(0)=0$ and $P(1)=1$. Let also $f(x)=[P'(x)]^2-20xP'(x)$. Find the smallest possible value of $\int_0^1 f(x)\,dx$.

全卷完

END OF PAPER