第十四屆培正數學邀請賽（**2015** 年）

14th Pui Ching Invitational Mathematics Competition (2015)

決賽（中四組）

Final Event (Secondary 4)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

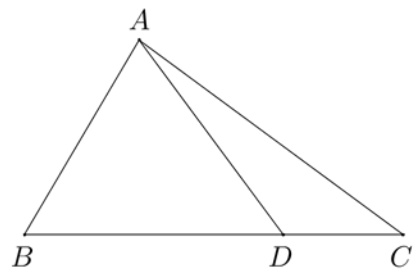
Questions 1 to 4 each carries 3 marks.

1. 設 $a_1, a_2, \ldots, a_{2015}$ 為等差數列。若 $a_3 + a_{100} = 212$ 而 $a_5 + a_{2014} = 4044$，求 a_1 的值。

   Let $a_1, a_2, \ldots, a_{2015}$ be an arithmetic sequence. If $a_3 + a_{100} = 212$ and $a_5 + a_{2014} = 4044$, find the value of a_1.

2. 在 ΔABC 中，$BC = 10$。D 是 BC 上的一點，使得 $AD = 6$ 及 $\angle ADB = 60°$。求 ΔABC 的面積。

   In ΔABC, $BC = 10$. D is a point on BC such that $AD = 6$ and $\angle ADB = 60°$. Find the area of ΔABC.



3. 若 $a + b = c^5$，其中 a、b 是連續正奇數而 c 是整數，求 a 的最小可能值。

   If $a + b = c^5$, where a, b are consecutive positive odd numbers while c is an integer, find the smallest possible value of a.

4. 費波那契數列 1, 1, 2, 3, 5, 8, … 首 2015 項之和除以 7 時的餘數是多少？

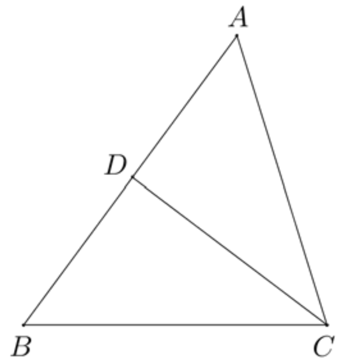
   What is the remainder when the sum of the first 2015 terms of the Fibonacci sequence 1, 1, 2, 3, 5, 8, … is divided by 7?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 在 ΔABC 中，$AC=6$。D 是 AB 上的一點，使得 $AD=BD=CD=5$。求 BC 的長度。

   In ΔABC, $AC=6$. D is a point on AB such that $AD=BD=CD=5$. Find the length of BC.



6. 一個等邊三角形內接於一個面積為 1 的圓形。求等邊三角形的面積。

   An equilateral triangle is inscribed in a circle of area 1. Find the area of the equilateral triangle.

7. 設 α、β、γ 為方程 $x^3-20x-15=0$ 的三個根。求 $\alpha^2+\beta^2+\gamma^2$ 的值。

   Let α, β, γ be the three roots to the equation $x^3-20x-15=0$. Find the value of $\alpha^2+\beta^2+\gamma^2$.

8. 若方程 $x^2-100x+k=0$（其中 k 是整數）的兩根均為質數，k 有多少個不同的可能值？

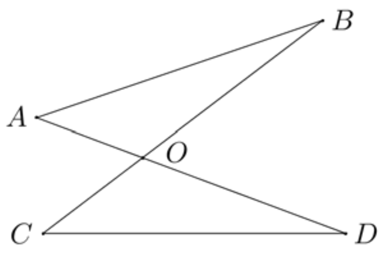
   If the two roots to the equation $x^2-100x+k=0$ (where k is an integer) are both prime numbers, how many different possible values of k are there?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 圖中，AOD 和 BOC 為直線，$OA=OC=4$，$OB=10$，$OD=5$，$CD=8$。求 AB 的長度。

   In the figure, AOD and BOC are straight lines, $OA=OC=4$, $OB=10$, $OD=5$ and $CD=8$. Find the length of AB.



10. 若 $a=-105$ 和 $b=209$ ，求 $2a^3+30a^2+156a+b^3-27b^2+243b-449$ 的值。

If $a=-105$ and $b=209$, find the value of $2a^3+30a^2+156a+b^3-27b^2+243b-449$.

11. 設 y 為負實數。若不等式 $\sin^2 x+y^2-1\geq(1-y)\cos x$ 對任意實數 x 均成立，求 y 的最大可能值。

Let y be a negative real number. If $\sin^2 x+y^2-1\geq(1-y)\cos x$ holds for any real number x, find the greatest possible value of y.

12. 若 $\log_2(-3\sin x°)=2\log_2(\cos x°)+1$ ，其中 $0\leq x\leq 360$ ，求 x 的所有可能值之和。

If $\log_2(-3\sin x°)=2\log_2(\cos x°)+1$ where $0\leq x\leq 360$, find the sum of all possible values of x.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 一個袋子中有 12 個球，包括 3 個白球、4 個紅球和 5 個綠球。若從袋中隨機抽出 5 個球，當中最少有 3 個球同色的概率是多少？

There are 12 balls in a bag, including 3 white, 4 red and 5 green. If 5 balls are picked randomly from the bag, what is the probability that at least three of them are of the same colour?

14. 若 $ac+bd=0$ ，則座標平面上 (a,b) 和 (c,d) 兩點稱為「垂直對」。我們最多可在座標平面上選取多少點，當中每點的 x 座標和 y 座標均為 –10 和 10 之間的整數（包括 –10 和 10），且當中沒有「垂直對」？

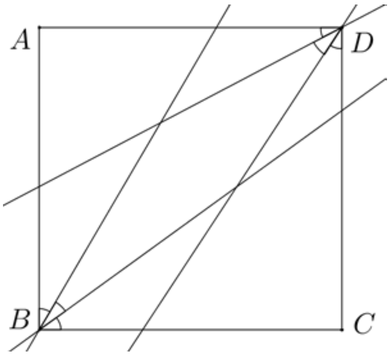
Two points (a,b) and (c,d) on the coordinate plane are said to be an ‘orthogonal pair’ if $ac+bd=0$. What is the maximum number of points that can be chosen on the coordinate plane so that each x-coordinate and each y-coordinate is an integer between –10 and 10 (inclusive), and that there is no ‘orthogonal pair’?

15. 求方程 $(x^2+1)^3+128x^3=48x^2(x^2+1)$ 的最小實根。

Find the smallest real root of the equation $(x^2+1)^3+128x^3=48x^2(x^2+1)$.

16. *ABCD* 是邊長 1 的正方形。$\angle ABC$ 和 $\angle ADC$ 各自的兩條三等分線圍成一個四邊形。求該四邊形的面積。

ABCD is a square of side length 1. The two trisectors of each of $\angle ABC$ and $\angle ADC$ form a quadrilateral. Find the area of that quadrilateral.



第 17 至第 20 題，每題 7 分。

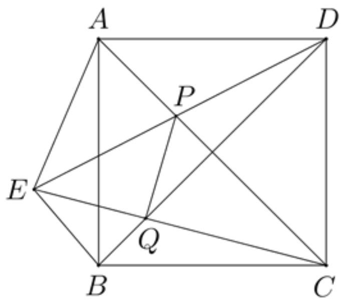
Questions 17 to 20 each carries 7 marks.

17. 在一個正方形內任意選取一點 *P*，並連接 *P* 和正方形的四個頂點，以把正方形分成四個三角形。求四個三角形的所有內角均不超過 $120°$ 的概率，答案準確至小數點後 2 位（提示：$\sqrt{3} \approx 1.73205$）。

A point *P* is chosen randomly in a square. Join *P* with the four vertices of the square so as to divide the square into four triangles. Find, correct to 2 decimal places, the probability that all interior angles of the four triangles do not exceed $120°$. (Hint: $\sqrt{3} \approx 1.73205$.)

18. 圖中，*ABCD* 是正方形，*E* 是正方形外的一點，使得 $\angle AEB = 135°$、$AE = 6$ 及 $EB = 4$。若 *DE* 和 *AC* 相交於 *P*，*CE* 和 *BD* 相交於 *Q*，求 *CDPQ* 的面積。

In the figure, *ABCD* is a square. *E* is a point outside the square such that $\angle AEB = 135°$, $AE = 6$ and $EB = 4$. If *DE* and *AC* intersect at *P* while *CE* and *BD* intersect at *Q*, find the area of *CDPQ*.



19. 求 $\tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} + \tan^{-1}\frac{2}{4^2} + \cdots$ 的值。

Find the value of $\tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} + \tan^{-1}\frac{2}{4^2} + \cdots$.

20. 在一個遊戲中，9 名編號為 1 至 9 的參賽者被分成三組：1、5、9 號參賽者一組，2、6、7 號參賽者一組，3、4、8 號參賽者一組。某袋子中裝有 1 至 9 號的球，然後把球逐一抽出，直至按以下規則得出一名勝利者為止：

- 若抽出的首三個球的編號總和大於 15，則第三個球的編號對應的參賽者勝出。
- 若抽出的首三個球的編號總和小於 15，則第四個球的編號對應的參賽者勝出。
- 若抽出的首三個球的編號總和等於 15，那麼
  - ➢ 若該三個編號對應三名同組的參賽者，則第三個球的編號對應的參賽者勝出；
  - ➢ 若該三個編號並非對應三名同組的參賽者，則第四個球的編號對應的參賽者勝出。

求 9 號參賽者勝出遊戲的概率。

In a game, there are 9 players, numbered 1 to 9. They are divided into three groups: Players 1, 5, 9 form a group; Players 2, 6, 7 form a group; Players 3, 4, 8 form a group. Nine balls numbered 1 to 9 are put into a bag, and drawn one by one until a winner is determined according to the following conditions:

- If the sum of the first three balls drawn is greater than 15, the winner is the one whose number is that on the third ball drawn.
- If the sum of the first three balls drawn is smaller than 15, the winner is the one whose number is that on the fourth ball drawn.
- If the sum of the first three balls drawn is equal to 15, then
  - ➢ if the three numbers correspond to three players from the same group, the winner is the one whose number is that on the third ball drawn;
  - ➢ if the three numbers do not correspond to three players from the same group, the winner is the one whose number is that on the fourth ball drawn.

Find the probability that Player 9 is the winner.

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END OF PAPER