第十四屆培正數學邀請賽（2015 年）

14th Pui Ching Invitational Mathematics Competition (2015)

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 若將一個正方形分成 25 個全等的小正方形，每個小正方形的周界是 224。若將原正方形分成 49 個全等的小正方形，則每個小正方形的周界是多少？

   If a square is divided into 25 identical smaller squares, each smaller square has perimeter 224. If the original square is divided into 49 identical smaller squares, what is the perimeter of each smaller square?

2. 若 $3a > b+14$ 及 $4b > a+23$，其中 a 和 b 為正整數，求 a 的最小可能值。

   If $3a > b+14$ and $4b > a+23$, where a and b be positive integers, find the smallest possible value of a.

3. 設 n 為三位平方數，其數字之和是 6 的倍數。求 n 的最小可能值。

   Let n be a three-digit square number whose sum of digits is a multiple of 6. Find the smallest possible value of n.

4. 在費波那契數列 1, 1, 2, 3, 5, 8, … 的首 2015 項中，有多少項是 7 的倍數？

   How many of the first 2015 terms of the Fibonacci sequence 1, 1, 2, 3, 5, 8, … are multiples of 7?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 求 $\dfrac{23^4+23^2+1}{3\times 23^2+4\times 23-20}$ 的值。

   Evaluate $\dfrac{23^4+23^2+1}{3\times 23^2+4\times 23-20}$.

6. 把一個凸 n 邊形沿一條直線剪開，可得到一個三角形和一個 2015 邊形。求 n 所有可能值之和。

When a convex n-sided polygon is cut along a straight line, one gets a triangle and a 2015-sided polygon. Find the sum of all possible values of n.

7. 如果某天的「月」和「日」相同，則那天稱為「好日子」，例如：1 月 1 日和 4 月 4 日都是「好日子」。那麼，同一年最多可以有多少個星期六是「好日子」？

A day is said to be 'good' if the 'month' and 'day' are the same. For example, 1st January and 4th April are both 'good' days. At most how many 'good' Saturdays can there be in the same year?

8. 一個袋子中有 10 個球，包括 5 個紅球、3 個綠球和 2 個藍球。若從袋中隨機抽出 7 個球，餘下 3 個球的顏色均不同的概率是多少？

There are 10 balls in a bag, including 5 red, 3 green and 2 blue. If 7 balls are picked randomly from the bag, what is the probability that the remaining 3 balls are all different in colour?

第 9 至第 12 題，每題 5 分。

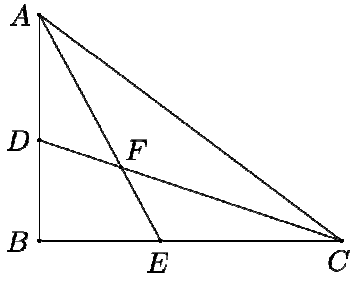
Questions 9 to 12 each carries 5 marks.

9. 若 x 為實數，求 $2015-8x-12x^2-8x^3-2x^4$ 的最大可能值。

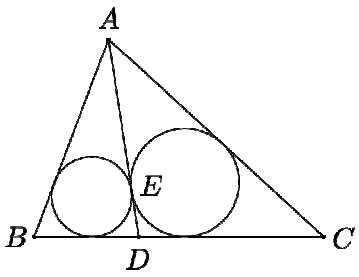
If x is a real number, find the greatest possible value of $2015-8x-12x^2-8x^3-2x^4$.

10. ABC 是直角三角形，B 是直角。D 和 E 分別是 AB 和 BC 上的點， F 是 AE 和 CD 的交點。若 $AD=10$ 、 $DB=6$ 、 $BE=8$ 及 $EC=16$ ，求 $DBEF$ 的面積。

ΔABC is right-angled at B. D and E are points on AB and BC respectively, and F is the intersection of AE and CD. If $AD=10$, $DB=6$, $BE=8$ and $EC=16$, find the area of $DBEF$.



11. 在 ΔABC 中，$AB=137$、$AC=241$ 及 $BC=200$。D 是 BC 上的一點，使得 ΔABD 和 ΔACD 的內切圓與 AD 相切於同一點 E。求 CD 的長度。



In ΔABC, $AB=137$, $AC=241$ and $BC=200$. D is a point on BC such that the inscribed circles of ΔABD and ΔACD touch AD at the same point E. Find the length of CD.

12. 如果某正整數由左至右和由右至左看皆相同，我們稱這個數為「回文數」，例如：3883、12321 和 25052 都是「回文數」。有多少個七位「回文數」由剛好兩個不同的數字組成（例如：2022202、3338333）？

If a positive integer reads the same from left to right as from right to left, it is called a 'palindrome'. For example, 3883, 12321 and 25052 are 'palindromes'. How many seven-digit 'palindromes' consist of exactly two different digits (e.g. 2022202, 3338333)?

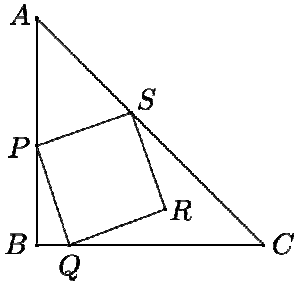
第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 給定 $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$，求 $1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\cdots$ 的值，其中分母為數列 1, 5, 7, 11, … 各項的平方，而該數列包含所有不能被 2 或 3 整除的正整數。

Given $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$. Find the value of $1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\cdots$, where the denominators are squares of the terms of the sequence 1, 5, 7, 11, …, which consist of all positive integers not divisible by 2 or 3.

14. 在 ΔABC 中，$AB=BC$ 及 $\angle ABC=90°$。P、Q 和 S 分別是 AB、BC 和 CA 上的點，R 是 ΔABC 內的一點，使得 $PQRS$ 成一正方形，其面積等於 ΔABC 的面積的 $\frac{2}{5}$。求 $\frac{BP}{BQ}$ 的值。



In ΔABC, $AB=BC$ and $\angle ABC=90°$. P, Q and S are points on AB, BC and CA respectively and R is a point inside ΔABC such that $PQRS$ is a square whose area is $\frac{2}{5}$ that of ΔABC. Find the value of $\frac{BP}{BQ}$.

15. 若 a、b、c、d 為正整數，使得 $ab=cd<1000$ 及 $a+b=c+d+1$，求 ab 的最大可能值。

Let a, b, c, d be positive integers such that $ab=cd<1000$ and $a+b=c+d+1$. Find the greatest possible value of ab.

16. 設 p 為質數，a 和 b 為正整數，使得 $\frac{p}{73}=\frac{1}{a}+\frac{1}{b}$。求 a 的所有可能值之和。

Let p be a prime number and a, b be positive integers such that $\frac{p}{73}=\frac{1}{a}+\frac{1}{b}$. Find the sum of all possible values of a.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 一個球體內切於一個正方錐體。若該錐體各邊的長度均為 1，求該球體的半徑。

A sphere is inscribed in a square pyramid, all of whose edges have length 1. Find the radius of the sphere.

18. 設 $ABCD$ 為正四面體，E 為 D 沿平面 ABC 反射後所得的點。已知直線 AD 與平面 ABC 的夾角為 54.7°（準確至小數點後一位），直線 AD 與平面 EBC 的夾角為 $x°$，其中 $0<x<90$。求 x，答案準確至最接近整數。

Let $ABCD$ be a regular tetrahedron, and let E be the point obtained by reflecting D across plane ABC. Given the angle between line AD and plane ABC is 54.7° (correct to one decimal place) and the angle between line AD and plane EBC is $x°$, where $0<x<90$. Find x correct to the nearest integer.

19. 如果某正整數由左至右和由右至左看皆相同，我們稱這個數為「回文數」，例如：3883、12321 和 25052 都是「回文數」。有多少種方法從 1 至 2015 當中選取兩個不同的整數，使它們之和是一個四位「回文數」？

If a positive integer reads the same from left to right as from right to left, it is called a 'palindrome'. For example, 3883, 12321 and 25052 are 'palindromes'. How many ways are there to choose two distinct integers from 1 to 2015 such that their sum is a 4-digit 'palindrome'?

20. 在一個遊戲中，9 名編號為 1 至 9 的參賽者被分成三組：1、5、9 號參賽者一組，2、6、7 號參賽者一組，3、4、8 號參賽者一組。某袋子中裝有 1 至 9 號的球，然後把球逐一抽出，直至按以下規則得出一名勝利者為止：

- 若抽出的首三個球的編號總和大於 15，則第三個球的編號對應的參賽者勝出。
- 若抽出的首三個球的編號總和小於 15，則第四個球的編號對應的參賽者勝出。
- 若抽出的首三個球的編號總和等於 15，那麼
  - 若該三個編號對應三名同組的參賽者，則第三個球的編號對應的參賽者勝出；
  - 若該三個編號並非對應三名同組的參賽者，則第四個球的編號對應的參賽者勝出。

求 9 號參賽者勝出遊戲的概率。

In a game, there are 9 players, numbered 1 to 9. They are divided into three groups: Players 1, 5, 9 form a group; Players 2, 6, 7 form a group; Players 3, 4, 8 form a group. Nine balls numbered 1 to 9 are put into a bag, and drawn one by one until a winner is determined according to the following conditions:

- If the sum of the first three balls drawn is greater than 15, the winner is the one whose number is that on the third ball drawn.
- If the sum of the first three balls drawn is smaller than 15, the winner is the one whose number is that on the fourth ball drawn.
- If the sum of the first three balls drawn is equal to 15, then
  - if the three numbers correspond to three players from the same group, the winner is the one whose number is that on the third ball drawn;
  - if the three numbers do not correspond to three players from the same group, the winner is the one whose number is that on the fourth ball drawn.

Find the probability that Player 9 is the winner.

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END OF PAPER