第十四屆培正數學邀請賽（2015 年）

14th Pui Ching Invitational Mathematics Competition (2015)

決賽（中二組）

Final Event (Secondary 2)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

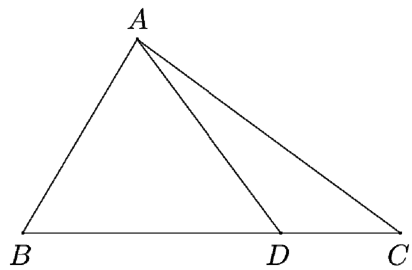
Questions 1 to 4 each carries 3 marks.

1. 某班 60 名學生排成一行，每個回合他們由左至右從 1 開始報數，報出 5 的倍數的學生需離開，然後重複進行下一回合，直至餘下的學生數目小於 5 為止。那麼，這個過程需經過多少個回合後才會停止？

   The 60 students of a class are aligned in a row. In each round, they count from 1 onwards from left to right. The students counting multiples of 5 have to leave. This process is then repeated in the next round and so on, until the number of remaining students is less than 5. How many rounds will this process last before it comes to a stop?

2. 在 ΔABC 中，$BC=10$。D 是 BC 上的一點，使得 $AD=6$ 及 $\angle ADB=60°$。求 ΔABC 的面積。

   In ΔABC, $BC=10$. D is a point on BC such that $AD=6$ and $\angle ADB=60°$. Find the area of ΔABC.



3. 若 $3a>b+14$ 及 $4b>a+23$，其中 a 和 b 為正整數，求 a 的最小可能值。

   If $3a>b+14$ and $4b>a+23$, where a and b be positive integers, find the smallest possible value of a.

4. 費波那契數列 1, 1, 2, 3, 5, 8, … 的第 2015 項除以 7 時的餘數是多少？

   In the Fibonacci sequence 1, 1, 2, 3, 5, 8, … , what is the remainder when the 2015th term is divided by 7?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 現有 2015 個整數，每個均等於 1 或 -1。這些數之和有多少個不同可能值？

   There are 2015 integers, each being either 1 or -1. How many different possible values are there for the sum of these numbers?

6. 在一個正十邊形中，有多少條對角線可把它分成兩個多邊形，其中每個多邊形的邊長數目均大於 3？

   How many diagonals in a regular decagon have the property that it divides the decagon into two polygons, each having more than 3 sides?

7. 求 $\dfrac{23^4+23^2+1}{3\times 23^2+4\times 23-20}$ 的值。

   Evaluate $\dfrac{23^4+23^2+1}{3\times 23^2+4\times 23-20}$.

8. 把一個凸 n 邊形沿一條直線剪開，可得到一個三角形和一個 2015 邊形。求 n 所有可能值之和。

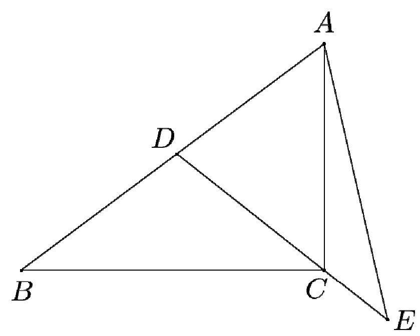
   When a convex n-sided polygon is cut along a straight line, one gets a triangle and a 2015-sided polygon. Find the sum of all possible values of n.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. ABC 是直角三角形，C 是直角。D 是 AB 的中點，E 則是 DC 延線上的一點，使得 $AE=DE$。若 $\angle AED=24°$ 而 $\angle BAC=x°$，求 x 的值。

   ΔABC is right-angled at C. D is the mid-point of AB, and E is a point on DC produced such that $AE=DE$. If $\angle AED=24°$ and $\angle BAC=x°$, find the value of x.



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When trying to calculate the product of two two-digit positive integers m and n, Lily mistakenly swaps the two digits of n. She ends up getting an answer which is 999 larger than the correct answer. Find m.

11. 在首 1200 個正整數中，有多少個最少有兩個相同的數字（例如：55、333、747）？

How many of the first 1200 positive integers have at least two digits being the same (e.g. 55, 333, 747)?

12. 在 ΔABC 中，$AB = 137$ 、 $AC = 241$ 及 $BC = 200$ 。D 是 BC 上的一點，使得 ΔABD 和 ΔACD 的內切圓與 AD 相切於同一點 E。求 CD 的長度。

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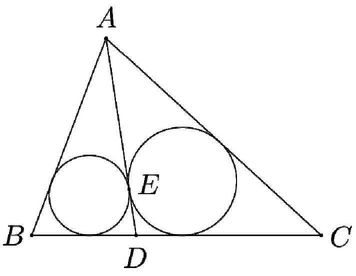
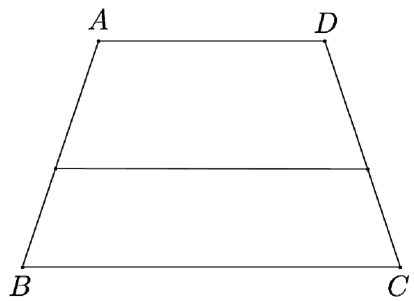


第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 如圖所示，$ABCD$ 是等腰梯形，其中 AD // BC、$AD = 20$ 、 $BC = 52$ 及 $AB = CD = 34$ 。一條平行於底邊的直線將梯形分成兩部分，使得上半部與下半部的面積比為 $2:1$。求這條直線與上底 AD 之間的距離。

In the figure, $ABCD$ is an isosceles trapezium with AD // BC, $AD = 20$, $BC = 52$ and $AB = CD = 34$. A straight line parallel to the bases divides the trapezium into two parts. If the ratio of areas of the upper part to the lower part is $2:1$, find the distance between this line and the upper base AD.



14. 給定 $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$，求 $1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\cdots$ 的值，其中分母為數列 1, 5, 7, 11, … 各項的平方，而該數列包含所有不能被 2 或 3 整除的正整數。

Given $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$. Find the value of $1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\cdots$, where the denominators are squares of the terms of the sequence 1, 5, 7, 11, …, which consist of all positive integers not divisible by 2 or 3.

15. 在一個 4×4 方格表的遊戲中，開始時每個方格內有一個金幣。在每個回合，參賽者需選取方格表中有超過一個金幣的一行或一列，然後把該行或該列的所有金幣取走，再在該行或該列的其中一格放回一個金幣。以上過程不斷重複，直至無法按此規則繼續為止。遊戲最多可以進行多少個回合？

A game is played on a 4×4 grid as follows. At the beginning, one coin is placed inside each cell. In each round, the participant has to select a row or a column containing more than one coin, then take away all coins in that row or column, and finally put back one coin in a cell of that row or column. This is repeated until it is no longer possible to continue according to the above rules. What is the maximum number of rounds that the game can last for?

16. 設 p 為質數，a 和 b 為正整數，使得 $\frac{p}{73}=\frac{1}{a}+\frac{1}{b}$。求 a 的所有可能值之和。

Let p be a prime number and a, b be positive integers such that $\frac{p}{73}=\frac{1}{a}+\frac{1}{b}$. Find the sum of all possible values of a.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 a、b、c 為正整數，它們分別可被 3、5、7 整除。若 $a+b$、$b+c$ 和 $c+a$ 分別可被 67、17 和 73 整除，求 $a+b+c$ 的最小可能值。

Let a, b and c be positive integers that are divisible by 3, 5, 7 respectively. If $a+b$, $b+c$ and $c+a$ are divisible by 67, 17 and 73 respectively, find the smallest possible value of $a+b+c$.

18. 一個球體內切於一個正方錐體。若該錐體各邊的長度均為 1，求該球體的半徑。

A sphere is inscribed in a square pyramid, all of whose edges have length 1. Find the radius of the sphere.

19. 設 $ABCD$ 為正四面體，E 為 D 沿平面 ABC 反射後所得的點。已知直線 AD 與平面 ABC 的夾角為 $54.7°$（準確至小數點後一位），直線 AD 與平面 EBC 的夾角為 $x°$，其中 $0 < x < 90$。求 x，答案準確至最接近整數。

Let $ABCD$ be a regular tetrahedron, and let E be the point obtained by reflecting D across plane ABC. Given the angle between line AD and plane ABC is $54.7°$ (correct to one decimal place) and the angle between line AD and plane EBC is $x°$, where $0 < x < 90$. Find x correct to the nearest integer.

20. 某國家有 A、B、C、D、E 五個城市，原先每兩個城市之間均有一條由中央政府管理的雙程路連接。現在政府決定將其中 6 條路改成單程路，每個城市均負責管理從該城市出發的所有單程路。已知在改動後，從每個城市均可只使用雙程路到達任意另一城市，而 A、B、C、D、E 五個城市所管理的單程路數目分別為 a、b、c、d 和 e。那麼，(a, b, c, d, e) 有多少組可能值？

There are 5 cities, A, B, C, D, E, in a country. Originally every pair of cities is connected by a bidirectional road managed by the central government. Now the government decides to change 6 of the roads to become unidirectional, and each city shall manage the unidirectional roads going out of the city. It is known that after the change, one can still travel from any city to any other city using bidirectional roads only, and that the number of roads managed by cities A, B, C, D, E are a, b, c, d and e respectively. How many sets of possible values of (a, b, c, d, e) are there?

全卷完

END OF PAPER

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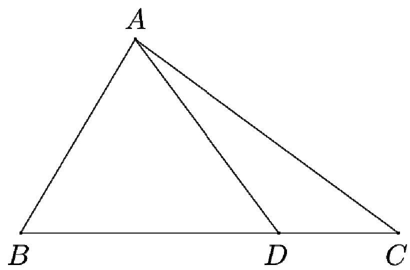

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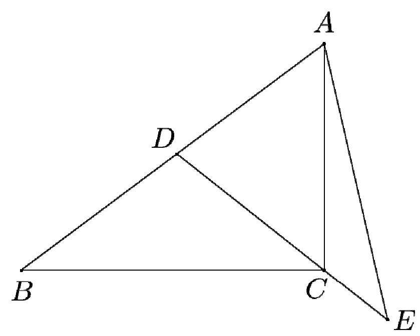

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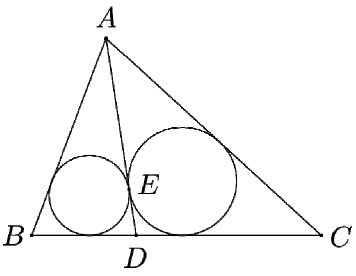

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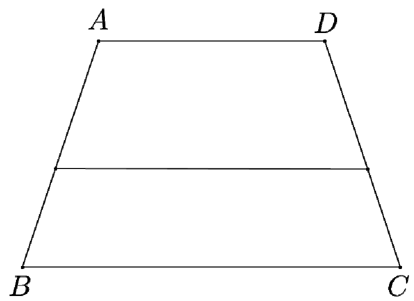

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Let *ABCD* be a regular tetrahedron, and let *E* be the point obtained by reflecting *D* across plane *ABC*. Given the angle between line *AD* and plane *ABC* is 54.7° (correct to one decimal place) and the angle between line *AD* and plane *EBC* is $x°$, where $0 < x < 90$. Find x correct to the nearest integer.

20. 某國家有 *A*、*B*、*C*、*D*、*E* 五個城市，原先每兩個城市之間均有一條由中央政府管理的雙程路連接。現在政府決定將其中 6 條路改成單程路，每個城市均負責管理從該城市出發的所有單程路。已知在改動後，從每個城市均可只使用雙程路到達任意另一城市，而 *A*、*B*、*C*、*D*、*E* 五個城市所管理的單程路數目分別為 *a*、*b*、*c*、*d* 和 *e*。那麼，(a, b, c, d, e) 有多少組可能值？

There are 5 cities, *A*, *B*, *C*, *D*, *E*, in a country. Originally every pair of cities is connected by a bidirectional road managed by the central government. Now the government decides to change 6 of the roads to become unidirectional, and each city shall manage the unidirectional roads going out of the city. It is known that after the change, one can still travel from any city to any other city using bidirectional roads only, and that the number of roads managed by cities *A*, *B*, *C*, *D*, *E* are *a*, *b*, *c*, *d* and *e* respectively. How many sets of possible values of (a, b, c, d, e) are there?

全卷完

END OF PAPER