第十四屆培正數學邀請賽（2015 年）

14th Pui Ching Invitational Mathematics Competition (2015)

決賽（中一組）

Final Event (Secondary 1)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 若將一個正方形分成 25 個全等的小正方形，每個小正方形的周界是 224。若將原正方形分成 49 個全等的小正方形，則每個小正方形的周界是多少？

   If a square is divided into 25 identical smaller squares, each smaller square has perimeter 224. If the original square is divided into 49 identical smaller squares, what is the perimeter of each smaller square?

2. 一組 40 人來到河岸邊，岸邊有一艘最多能載四人的小船，小船必須由一人駕駛。那麼最少需要駕駛多少次才可讓所有人過河（小船由河的一邊駛至另一邊時算作一次）？

   A group of 40 people arrive at a river bank where there is a single boat that can carry a maximum of four people. The boat has to be operated by one person. What is the minimum number of boat trips needed for all the people to cross the river? (When the boat travels from one side of the river to the other side it is counted as one trip.)

3. 若 $a+b=c^5$，其中 a、b 是連續正奇數而 c 是整數，求 a 的最小可能值。

   If $a+b=c^5$, where a, b are consecutive positive odd numbers while c is an integer, find the smallest possible value of a.

4. 費波那契數列 1, 1, 2, 3, 5, 8, … 的第 2015 項除以 7 時的餘數是多少？

   In the Fibonacci sequence 1, 1, 2, 3, 5, 8, … , what is the remainder when the 2015th term is divided by 7?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 現有 2015 個整數，每個均等於 1 或 -1。這些數之和有多少個不同可能值？

   There are 2015 integers, each being either 1 or -1. How many different possible values are there for the sum of these numbers?

6. 在一個正十邊形中，有多少條對角線可把它分成兩個多邊形，其中每個多邊形的邊長數目均大於 3？

   How many diagonals in a regular decagon have the property that it divides the decagon into two polygons, each having more than 3 sides?

7. 如果某天的「月」和「日」相同，則那天稱為「好日子」，例如：1 月 1 日和 4 月 4 日都是「好日子」。那麼，同一年最多可以有多少個星期六是「好日子」？

   A day is said to be ‘good’ if the ‘month’ and ‘day’ are the same. For example, 1st January and 4th April are both ‘good’ days. At most how many ‘good’ Saturdays can there be in the same year?

8. 在一個數學比賽中，其中一道題目要求參賽者找出 $\frac{1007}{2015}$ 小數點後第 100000 個位。甲和乙是兩個參賽者，已知甲找出正確的答案，而乙把分數錯誤看成 $\frac{1008}{2015}$，那麼甲和乙的答案之和是甚麼？

   In a mathematical competition, one question asks contestants to find the 100000-th digit after the decimal point if $\frac{1007}{2015}$ is written as a decimal. Alice and Bob were two contestants for the competition, with Alice getting the correct answer but Bob misreading the fraction as $\frac{1008}{2015}$. What is the sum of Alice’s and Bob’s answers?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 若 $a=-105$ 和 $b=209$ ，求 $2a^3+30a^2+156a+b^3-27b^2+243b-449$ 的值。

   If $a=-105$ and $b=209$, find the value of $2a^3+30a^2+156a+b^3-27b^2+243b-449$.

10. 小莉嘗試計算兩個兩位正整數 m 和 n 的乘積時，誤把 n 的兩位數字調轉了，結果得出的乘積比正確答案大 999。求 m。

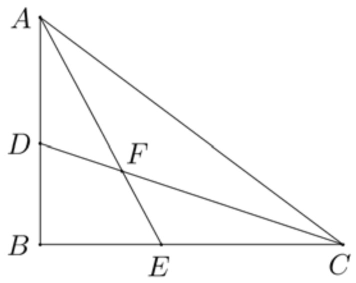
When trying to calculate the product of two two-digit positive integers m and n, Lily mistakenly swaps the two digits of n. She ends up getting an answer which is 999 larger than the correct answer. Find m.

11. 有多少個四位數由剛好兩個不同的數字組成（例如：2022、3338）？

How many four-digit numbers consist of exactly two different digits (e.g. 2022, 3338)?

12. ABC 是直角三角形，B 是直角。D 和 E 分別是 AB 和 BC 上的點，F 是 AE 和 CD 的交點。若 $AD=10$、$DB=6$、$BE=8$ 及 $EC=16$，求 $DBEF$ 的面積。

ΔABC is right-angled at B. D and E are points on AB and BC respectively, and F is the intersection of AE and CD. If $AD=10$, $DB=6$, $BE=8$ and $EC=16$, find the area of $DBEF$.

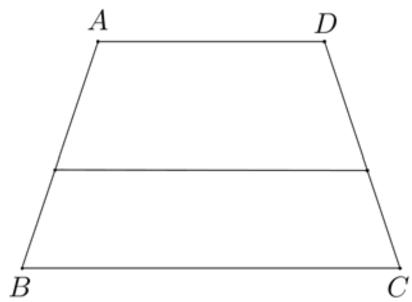


第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 如圖所示，$ABCD$ 是等腰梯形，其中 $AD // BC$、$AD=20$、$BC=52$ 及 $AB=CD=34$。一條平行於底邊的直線將梯形分成兩部分，使得上半部與下半部的面積比為 $2:1$。求這條直線與上底 AD 之間的距離。

In the figure, $ABCD$ is an isosceles trapezium with $AD // BC$, $AD=20$, $BC=52$ and $AB=CD=34$. A straight line parallel to the bases divides the trapezium into two parts. If the ratio of areas of the upper part to the lower part is $2:1$, find the distance between this line and the upper base AD.



14. 在一個 4×4 方格表的遊戲中，開始時每個方格內有一個金幣。在每個回合，參賽者需選取方格表中有超過一個金幣的一行或一列，然後把該行或該列的所有金幣取走，再在該行或該列的其中一格放回一個金幣。以上過程不斷重複，直至無法按此規則繼續為止。遊戲最多可以進行多少個回合？

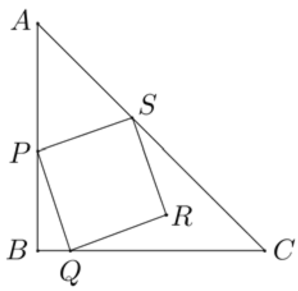
A game is played on a 4×4 grid as follows. At the beginning, one coin is placed inside each cell. In each round, the participant has to select a row or a column containing more than one coin, then take away all coins in that row or column, and finally put back one coin in a cell of that row or column. This is repeated until it is no longer possible to continue according to the above rules. What is the maximum number of rounds that the game can last for?

15. 求 $\frac{1}{5}+\frac{1}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\frac{5}{5^5}+\frac{8}{5^6}+\cdots$ 的值（其中各項的分子來自費波那契數列 1, 1, 2, 3, 5, 8, 13, 21, …）。

Find the value of $\frac{1}{5}+\frac{1}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\frac{5}{5^5}+\frac{8}{5^6}+\cdots$, where the numerators come from the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, …

16. 在 ΔABC 中，$AB=BC$ 及 $\angle ABC=90^\circ$。P、Q 和 S 分別是 AB、BC 和 CA 上的點，R 是 ΔABC 內的一點，使得 $PQRS$ 成一正方形，其面積等於 ΔABC 的面積的 $\frac{2}{5}$。求 $\frac{BP}{BQ}$ 的值。

In ΔABC, $AB=BC$ and $\angle ABC=90^\circ$. P, Q and S are points on AB, BC and CA respectively and R is a point inside ΔABC such that $PQRS$ is a square whose area is $\frac{2}{5}$ that of ΔABC. Find the value of $\frac{BP}{BQ}$.



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 a、b、c 為正整數，它們分別可被 3、5、7 整除。若 $a+b$、$b+c$ 和 $c+a$ 分別可被 67、17 和 73 整除，求 $a+b+c$ 的最小可能值。

Let a, b and c be positive integers that are divisible by 3, 5, 7 respectively. If $a+b$, $b+c$ and $c+a$ are divisible by 67, 17 and 73 respectively, find the smallest possible value of $a+b+c$.

18. 某國家有 A、B、C、D、E 五個城市，原先每兩個城市之間均有一條由中央政府管理的雙程路連接。現在政府決定將其中 6 條路改成單程路，每個城市均負責管理從該城市出發的所有單程路。已知在改動後，從每個城市均可只使用雙程路到達任意另一城市，而 A、B、C、D、E 五個城市所管理的單程路數目分別為 a、b、c、d 和 e。那麼，(a, b, c, d, e) 有多少組可能值？

There are 5 cities, A, B, C, D, E, in a country. Originally every pair of cities is connected by a bidirectional road managed by the central government. Now the government decides to change 6 of the roads to become unidirectional, and each city shall manage the unidirectional roads going out of the city. It is known that after the change, one can still travel from any city to any other city using bidirectional roads only, and that the number of roads managed by cities A, B, C, D, E are a, b, c, d and e respectively. How many sets of possible values of (a, b, c, d, e) are there?

19. 已知 $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$ 及 $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{13^2}+\cdots=\frac{15}{\pi^2}$（後者的分母為數列 1, 2, 3, 5, 6, 7, 10, …各項的平方，而該數列包含所有除 1 以外沒有其他因數是平方數的正整數）。求 $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots$ 的值。

Given $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$ and $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{13^2}+\cdots=\frac{15}{\pi^2}$, where the denominators in the latter sum are squares of the terms of the sequence 1, 2, 3, 5, 6, 7, 10, …, which consist of all positive integers without square factors except 1. Find the value of $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots$.

20. 如果某正整數由左至右和由右至左看皆相同，我們稱這個數為「回文數」，例如：3883、12321 和 25052 都是「回文數」。有多少種方法從 1 至 2015 當中選取兩個不同的整數，使它們之和是一個四位「回文數」？

If a positive integer reads the same from left to right as from right to left, it is called a ‘palindrome’. For example, 3883, 12321 and 25052 are ‘palindromes’. How many ways are there to choose two distinct integers from 1 to 2015 such that their sum is a 4-digit ‘palindrome’?

全卷完

END OF PAPER