第十三屆培正數學邀請賽（**2014** 年）

13th Pui Ching Invitational Mathematics Competition (2014)

初賽（高中組）

Heat Event (Senior Secondary)

時限：**1** 小時 **15** 分

Time allowed: 1 hour 15 minutes

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 作答時，每題的答案均須以 0 至 9999 之間的整數表示。依照答題紙上的指示填寫答案，毋須呈交計算步驟。

Each answer must be given in the form of an integer between 0 and 9999. Follow the instructions on the answer sheet to enter the answers. You are not required to hand in your steps of working.

(d) 不得使用計算機。

The use of calculators is not allowed.

(e) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：每題的答案均須以 0 至 9999 之間的整數表示，如有需要應以上述範圍內最接近正確答案的整數回答。如有兩個這樣的整數與正確答案同樣接近，則以「四捨五入」的原則取較大的整數。請細閱答題紙上的指示。

Note: Each answer must be given in the form of an integer between 0 and 9999. Where necessary, the answer should be rounded off to the nearest integer in the above range. Read the instructions on the answer sheet in detail.

1. 在小於 100 的質數中，有多少個的個位數字為 7？ （3 分）

How many prime numbers smaller than 100 are there whose unit digits are 7? (3 marks)

2. 設 $n=1+2+\cdots+k$（其中 k 是正整數），且 n 最少有 4 個不同質因數。求 n 的最小可能值。 （3 分）

Let $n=1+2+\cdots+k$, where k is a positive integer, such that n has at least 4 distinct prime factors. Find the smallest possible value of n. (3 marks)

3. 設 $a_1=\sqrt{2}$，並對正整數 n 定義 $a_{n+1}=\sqrt{2a_n}$。若 $a_8=2^{\frac{p}{q}}$，其中 $\frac{p}{q}$ 為最簡分數。求 p 的值。 （3 分）

Let $a_1=\sqrt{2}$ and define $a_{n+1}=\sqrt{2a_n}$ for positive integers n. If $a_8=2^{\frac{p}{q}}$ where $\frac{p}{q}$ is in lowest form, find the value of p. (3 marks)

4. 兩名網球員進行 5 局比賽，先取得 3 局勝利的一方立即獲勝。若每局比賽中雙方獲勝的概率相等而且沒有平局，則需要進行第 5 局比賽才可分出勝負的概率以最簡分數表示時為 $\frac{a}{b}$。求 $a+b$ 的值。 （3 分）

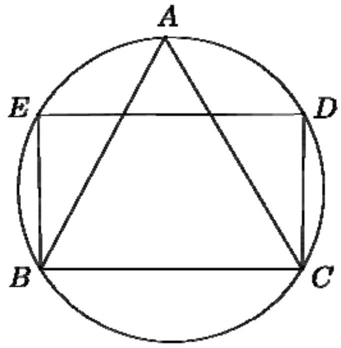
Two tennis players compete in a match consisting of 5 games. The one who first wins 3 games will win the match immediately. If the two sides have equal probability to win each game and no draw is possible in a game, then the probability of requiring to play the fifth game to determine the winner is $\frac{a}{b}$ in lowest form. Find the value of $a+b$. (3 marks)

5. 已知 100 有 9 個正因數，記它們為 d_1、d_2、…、d_9。求 $\log\frac{100}{d_1}+\log\frac{100}{d_2}+\cdots+\log\frac{100}{d_9}$ 的值。 （4 分）

It is known that 100 has 9 positive factors. Denote them by d_1, d_2, …, d_9. Find the value of $\log\frac{100}{d_1}+\log\frac{100}{d_2}+\cdots+\log\frac{100}{d_9}$. (4 marks)

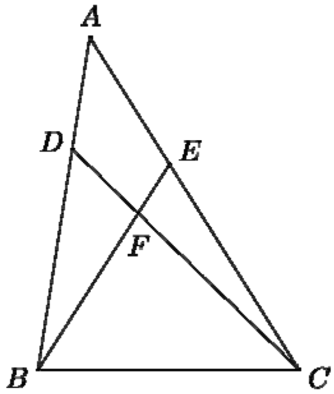
6. 如圖所示，等邊三角形 ABC 內接於一個半徑爲 4 的圓形。設 D 和 E 爲圓上的點，使得 $BCDE$ 爲長方形。若 $BCDE$ 面積爲 S，求 S^2 的值。（4 分）

In the figure, ΔABC is equilateral and inscribed in a circle of radius 4. Let D and E be points on the circle such that $BCDE$ is a rectangle. If the area of $BCDE$ is S, find the value of S^2. (4 marks)



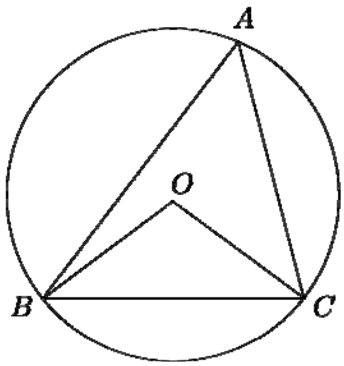
7. 在 ΔABC 中，BC 爲最短邊。設 D 和 E 分別爲 AB 和 AC 上的點，使得 $DB = BC = CE$。BE 和 CD 相交於 F。若 $\angle BAC = 38°$，且 $\angle DFE = x°$，求 x 的值。（4 分）

In ΔABC, BC is the shortest side. Let D and E be points on AB and AC respectively such that $DB = BC = CE$. BE and CD intersect at F. If $\angle BAC = 38°$ and $\angle DFE = x°$, find the value of x. (4 marks)



8. 如圖所示，O 是 ΔABC 的外心。若 $\angle BAC = 62°$、$\angle OBC = 2\angle ABO$ 及 $\angle ACO = x°$，求 x 的值。（4 分）

In the figure, O is the circumcentre of ΔABC. If $\angle BAC = 62°$, $\angle OBC = 2\angle ABO$ and $\angle ACO = x°$, find the value of x. (4 marks)



9. 最少需要選取多少個兩位數，才可確保當中最少有 3 個的數字之和相等？（5 分）

At least how many two-digit numbers have to be chosen to ensure that at least 3 of the chosen numbers have the same sum of digits? (5 marks)

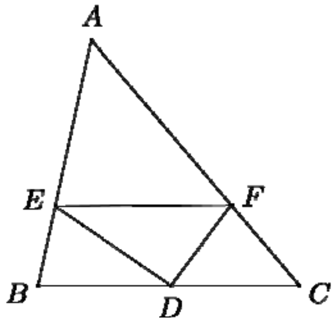
10. 設 N 表示擲出兩顆勻稱的骰子時較大點數的骰子上所標示的數字。若把 N 的期望值以最簡分數表示，其分子是甚麼？（5 分）

Let N denote the larger of the two numbers obtained when two fair dice are thrown. When the expected value of N is expressed as a fraction in lowest form, what is the numerator? (5 marks)

11. 已知 m、n 爲正整數，它們之和爲 252。若它們的最大公因數和最小公倍數之和爲 468，最大公因數和最小公倍數之差爲 396，求 m 和 n 中較大的一個數。（5分）

Let m and n be positive integers with sum 252. If the sum of their H.C.F. and their L.C.M. is 468 while the difference between their H.C.F. and their L.C.M. is 396, find the larger number among m and n. (5 marks)

12. 在 ΔABC 中，D 是 BC 的中點。E 和 F 分別是 AB 和 AC 上的點，使得 EF // BC。若 ΔAEF 和 ΔDEF 的面積比爲 $7:3$，且 ΔABC 的面積爲 100，求 ΔDFC 的面積。（5分）



In ΔABC, D is the mid-point of BC. E and F be points on AB and AC respectively such that EF // BC. If the ratio of the areas of ΔAEF and ΔDEF is $7:3$, and the area of ΔABC is 100, find the area of ΔDFC. (5 marks)

13. 方程 $x^2+nx+144=0$（其中 n 爲正整數）的兩個根均爲整數。那麼，n 有多少個不同的可能值？（5分）

Both roots to the equation $x^2+nx+144=0$ (where n is a positive integer) are integers. How many different possible values of n are there? (5 marks)

14. 設 a、b 爲兩個連續的平方數，其中 $a<b$。若 a^3+1 是 $b-a$ 的倍數，求 a 的最大可能值。（6分）

Let a and b be two consecutive square numbers with $a<b$. If a^3+1 is a multiple of $b-a$, find the greatest possible value of a. (6 marks)

15. 設 x 和 y 爲實數。若 $x^2+y^2=2x-4y+4$，求 $8x+15y$ 的最大可能值。（6分）

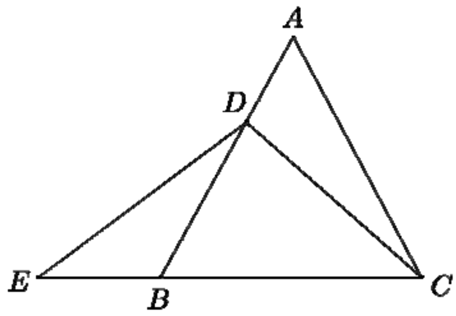
Let x and y be real numbers. If $x^2+y^2=2x-4y+4$, find the greatest possible value of $8x+15y$. (6 marks)

16. 已知對任意實數 x 皆有 $f(f(x))=20x-2014$，且存在實數 k 使得 $f(k)=20k-2014$。求 k 的值。（7分）

It is known that $f(f(x))=20x-2014$ for any real number x, and that there exists a real number k for which $f(k)=20k-2014$. Find the value of k. (7 marks)

17. 在 ΔABC 中，$AB = AC$。D 是 AB 上的一點，使得 $AD : DB = 2 : 5$。設 E 爲 CB 延長線上的一點，使得 $DE = DC$。若 ΔDEB 的面積爲 10，求 ΔABC 的面積。（7 分）

In ΔABC, $AB = AC$. D is a point on AB such that $AD : DB = 2 : 5$. Let E be a point on the extension of CB such that $DE = DC$. If the area of ΔDEB is 10, find the area of ΔABC. (7 marks)



18. 如圖所示，一個 3×3 的方格表的其中三格已填上整數。有多少種方法可以在餘下的方格內各填上一個整數（可以相同），使得每個方格內的數字均大於其上方及左方的數字？（7 分）

As shown in the figure, integers have been filled into three cells of a 3×3 table. How many ways are there to fill an integer (not necessarily distinct) into each of the remaining cells, so that the number in each cell is larger than the number on its top and on its left? (7 marks)

1		
	6	
		11

19. 對任意實數 x，定義 $f(x) = \cos x + 2\cos 2x + 3\cos 3x + \cdots + 2014\cos 2014x$。求 $\int_{-\pi}^{\pi} f(t)\cos 100t\, dt$ 的值。（7 分）

For a real number x, define $f(x) = \cos x + 2\cos 2x + 3\cos 3x + \cdots + 2014\cos 2014x$. Find the value of $\int_{-\pi}^{\pi} f(t)\cos 100t\, dt$. (7 marks)

20. 若 C_{1007}^{2014} 可被 10^k 整除（其中 k 爲正整數），求 k 的最大可能值。（7 分）

If C_{1007}^{2014} is divisible by 10^k, where k is a positive integer, find the greatest possible value of k. (7 marks)

全卷完

END OF PAPER