第十三屆培正數學邀請賽（2014年）

13th Pui Ching Invitational Mathematics Competition (2014)

決賽（高中組）

Final Event (Senior Secondary)

時限：2小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設20題，總分爲100分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 有多少個四位正整數的千位和百位相同，且十位和個位相同？

   How many four-digit positive integers are there such that the thousands and hundreds digits are equal, and the tens and unit digits are equal?

2. 某等差數列共有 2014 項，其中每項均是整數。若第 1000 項爲 2014，那麼這個數列中最多有多少項是 3 的倍數？

   There is an arithmetic sequence with 2014 terms in which every term is an integer. If the 1000th term is 2014, at most how many terms of this sequence can be multiples of 3?

3. 某等比數列共有 2014 項，且其首項和末項都不是整數。那麼這個數列中，最多有多少項是整數？

   A geometric sequence consists of 2014 terms and neither the first term nor the last term is an integer. What is the maximum number of integer terms in the sequence?

4. 設 $P(x)$ 爲二次多項式。若 $P(1)=1$、$P(2)=8$ 而 $P(3)=27$，求 $P(5)$ 的值。

   Let $P(x)$ be a quadratic polynomial. If $P(1)=1$, $P(2)=8$ and $P(3)=27$, find the value of $P(5)$.

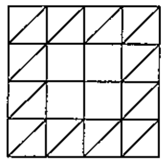
第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 若 $a+\frac{3}{a}=5$，求 $\frac{(a+2)^2-1}{a^3(5a-22)}$ 的值。

   If $a+\frac{3}{a}=5$, find the value of $\frac{(a+2)^2-1}{a^3(5a-22)}$.

6. 設 n 爲大於 1 的整數。現取一個 $n\times n$ 的正方形，先把它分成 n^2 個小正方形（面積各爲 1），再在邊界上的小正方形各畫一條對角線以分成兩個小三角形（面積各爲 0.5）。附圖顯示了 $n=4$ 時的情況，可見原正方形被分成 4 個小正方形和 24 個小三角形。若要使小正方形和小三角形的數目之差最小，則 n 的值應是甚麼？

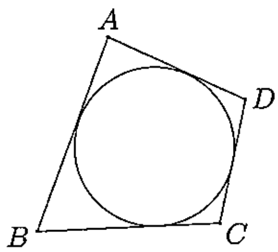


   Let n be an integer greater than 1. Now take an $n\times n$ square and divide it into n^2 small squares (each with area 1), and then split each small square on the boundary into two small triangles (each with area 0.5) by drawing a diagonal. The figure shows the situation for $n=4$ in which the original square is divided into 4 small squares and 24 small triangles. If the difference between the number of small squares and the number of small triangles is to be minimised, what should the value of n be?

7. 已知方程 $x^2+ax+b=0$（其中 a、b 爲不超過 10 的正整數）有兩個相異的實根。那麼 (a,b) 有多少組不同的可能値？

   It is given that the equation $x^2+ax+b=0$ (where a, b are positive integers not exceeding 10) has two distinct real roots. How many different sets of possible values of (a,b) are there?

8. 一個圓形內接於四邊形 $ABCD$，其中 $ABCD$ 的周界和面積分別爲 600 和 2014。求該圓形的半徑。



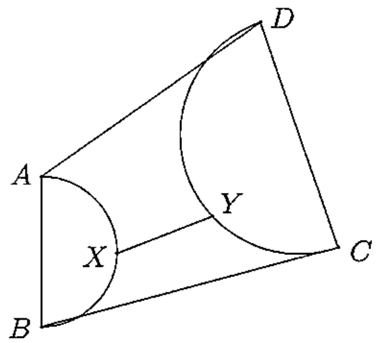
   A circle is inscribed in a quadrilateral $ABCD$, where the perimeter and area of $ABCD$ are 600 and 2014 respectively. Find the radius of the circle.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 四邊形 $ABCD$ 的頂點的座標分別爲 $A=(0,6)$、$B=(0,0)$、$C=(14,3)$ 和 $D=(10,13)$。以 AB 和 CD 爲直徑向四邊形內作半圓形。若 X 和 Y 分別在兩個半圓形上，求 X 和 Y 之間距離的最小可能值。

The coordinates of the vertices of a quadrilateral $ABCD$ are $A=(0,6)$, $B=(0,0)$, $C=(14,3)$ and $D=(10,13)$ respectively. Construct semi-circles towards the interior of the quadrilateral with diameters AB and CD. If X and Y are points on the two semi-circles respectively, find the smallest possible value of the distance between X and Y.



10. 小莉和小美進行乒乓球比賽，先取得 7 局勝利的一方爲優勝者。已知在首 6 局中，小莉勝出 4 局而小美勝出 2 局。若每局比賽中兩人獲勝的概率相同，且沒有平局，求小莉最後勝出的概率。

Lily and Mimi are having a table tennis competition. The one who wins 7 games first will become the winner. It is known that during the first 6 games, Lily has won 4 while Mimi has won 2. If they have the same chance of winning each game and there are no ties, find the probability for Lily to be the winner in the end.

11. 已知 $\log 2 \approx 0.30103$ 及 $\log 3 \approx 0.47712$。若 26^{100} 是個 n 位數，求 n 的值。

Given $\log 2 \approx 0.30103$ and $\log 3 \approx 0.47712$. If 26^{100} is an n-digit number, find the value of n.

12. 設 $a_1 = 2014$，並對正整數 n 定義 $a_{n+1} = a_1 + a_2 + \cdots + a_n + 1$。求 a_{2014} 的最後兩位數字。

Let $a_1 = 2014$ and define $a_{n+1} = a_1 + a_2 + \cdots + a_n + 1$ for positive integers n. Find the last two digits of a_{2014}.

第 13 至第 16 題，每題 6 分。

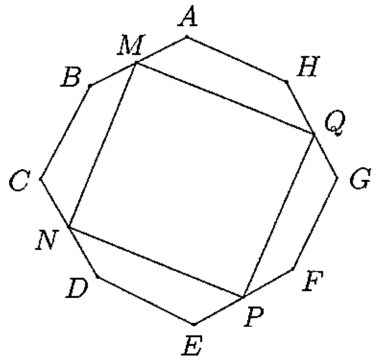
Questions 13 to 16 each carries 6 marks.

13. 若 m、n 爲正整數，且 $m^2-3n^2=9477$ ，求 m 的最小可能值。

 If m and n are positive integers such that $m^2-3n^2=9477$, find the smallest possible value of m.

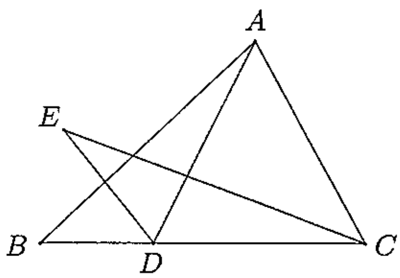
14. 圖中，$ABCDEFGH$ 是正八邊形，M、N、P、Q 分別是 AB、CD、EF 和 GH 的中點。若 $MNPQ$ 的面積爲 1，求 $ACEG$ 的面積，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 a、b 爲有理數。

 In the figure, $ABCDEFGH$ a regular octagon. M, N, P, Q are the mid-points of AB, CD, EF and GH respectively. If the area of $MNPQ$ is 1, find the area of $ACEG$, giving your answer in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where a and b are rational.



15. 如圖所示，D 是 ΔABC 邊 BC 上的一點。已知 $AD=AC=6$、$BD=3$ 及 $CD=5$。若 E 爲點 D 沿線段 AB 反射後所得的點，求 CE 的長度。

 As shown in the figure, D is a point on side BC of ΔABC. Suppose $AD=AC=6$, $BD=3$ and $CD=5$. If E is the point obtained by reflecting point D across line segment AB, find the length of CE.



16. 若 $10\sin\theta°+\cos 2\theta°=2\sin 3\theta°+2$ ，其中 $0<\theta<90$ ，求 $\tan\theta°$ 的值。

 If $10\sin\theta°+\cos 2\theta°=2\sin 3\theta°+2$ where $0<\theta<90$, find the value of $\tan\theta°$.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 a、b、c、d、e 爲整數，其中 (b, c) 和 (d, e) 爲拋物線 $y=ax^2+x$ 上的兩點。若這兩點的距離爲 99，求 $a^2-|b^2-d^2|$ 的值。

    Let a, b, c, d, e be integers such that (b, c) and (d, e) are two points on the parabola $y=ax^2+x$. If the distance between these two points is 99, find the value of $a^2-|b^2-d^2|$.

18. 設 $A=(0,1)$、$B=(4,9)$ 和 $C=(x,-1)$ 爲座標平面上的三點。求實數 x 的值，使得 $\angle ACB$ 達至最大值。

    Let $A=(0,1)$, $B=(4,9)$ and $C=(x,-1)$ be three points on the coordinate plane. Find the value of the real number x for which $\angle ACB$ attains its maximum value.

19. 求 $\int_0^\infty e^{-t^2}t^3dt$ 的值。

    Find the value of $\int_0^\infty e^{-t^2}t^3dt$.

20. 若 $\int_{-1}^{t}\left[\sin(x^3-9x^2+18x)+x^2-6x+\frac{11}{3}\right]dx=0$，求 t 的最大可能值。

    If $\int_{-1}^{t}\left[\sin(x^3-9x^2+18x)+x^2-6x+\frac{11}{3}\right]dx=0$, find the greatest possible value of t.

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END OF PAPER