# 第十三屆培正數學邀請賽（2014 年）

# 13th Pui Ching Invitational Mathematics Competition (2014)

## 決賽（中四組）

## Final Event (Secondary 4)

**時限：2 小時**

**Time allowed: 2 hours**

**參賽者須知：**

**Instructions to Contestants:**

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

**第 1 至第 4 題，每題 3 分。**

**Questions 1 to 4 each carries 3 marks.**

1. 設 $n$ 爲整數，使得 $\frac{38n-7}{20}<2n+1<\frac{24n-5}{13}$。求 $n$ 所有可能值之和。

   Let $n$ be an integer such that $\frac{38n-7}{20}<2n+1<\frac{24n-5}{13}$. Find the sum of all possible values of $n$.

2. 某等比數列共有 2014 項，且其首項和末項都不是整數。那麼這個數列中，最多有多少項是整數？

   A geometric sequence consists of 2014 terms and neither the first term nor the last term is an integer. What is the maximum number of integer terms in the sequence?

3. 設 $a_1$、$a_2$、…、$a_n$ 爲正整數，且 $a_1!+a_2!+\cdots+a_n!=2014$（其中 $m!=1\times2\times\cdots\times m$）。求 $n$ 的最小可能值。

   Let $a_1$, $a_2$, …, $a_n$ be positive integers such that $a_1!+a_2!+\cdots+a_n!=2014$ (where $m!=1\times2\times\cdots\times m$). Find the smallest possible value of $n$.

4. 某等差數列共有 2014 項，其中每項均是整數。若第 1000 項爲 2014，那麼這個數列中最多有多少項是 3 的倍數？

   There is an arithmetic sequence with 2014 terms in which every term is an integer. If the 1000th term is 2014, at most how many terms of this sequence can be multiples of 3?

**第 5 至第 8 題，每題 4 分。**

**Questions 5 to 8 each carries 4 marks.**

5. 設 $P(x)=x^3+ax^2+bx+c$，其中 $a$、$b$、$c$ 爲正整數。已知對任意正整數 $m$，$P(m)$ 均爲 $n$ 的倍數。求 $n$ 的最大可能值。

   Let $P(x)=x^3+ax^2+bx+c$, where $a$, $b$, $c$ are positive integers. It is given that for any positive integer $m$, $P(m)$ is a multiple of $n$. Find the greatest possible value of $n$.

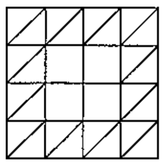
6. 現有五張分別寫上 0、1、1、2、3 的咭片。若把它們以不同的次序排成一行，共可得到多少個不同的五位正奇數？

   There are five cardboards with 0, 1, 1, 2, 3 written respectively. By arranging them in a row in different orders, how many different five-digit positive odd numbers can be formed?

7. 已知方程 $x^2+ax+b=0$（其中 $a$、$b$ 爲不超過 10 的正整數）有兩個相異的實根。那麼 $(a,b)$ 有多少組不同的可能值？

   It is given that the equation $x^2+ax+b=0$ (where $a$, $b$ are positive integers not exceeding 10) has two distinct real roots. How many different sets of possible values of $(a,b)$ are there?

8. 設 $n$ 爲大於 1 的整數。現取一個 $n \times n$ 的正方形，先把它分成 $n^2$ 個小正方形（面積各爲 1），再在邊界上的小正方形各畫一條對角線以分成兩個小三角形（面積各爲 0.5）。附圖顯示了 $n=4$ 時的情況，可見原正方形被分成 4 個小正方形和 24 個小三角形。若要使小正方形和小三角形的數目之差最小，則 $n$ 的值應是甚麼？

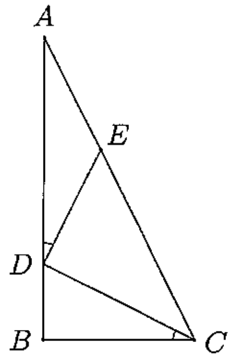
   Let $n$ be an integer greater than 1. Now take an $n \times n$ square and divide it into $n^2$ small squares (each with area 1), and then split each small square on the boundary into two small triangles (each with area 0.5) by drawing a diagonal. The figure shows the situation for $n=4$ in which the original square is divided into 4 small squares and 24 small triangles. If the difference between the number of small squares and the number of small triangles is to be minimised, what should the value of $n$ be?

**第 9 至第 12 題，每題 5 分。**

**Questions 9 to 12 each carries 5 marks.**

9. 在 $\Delta ABC$ 中，$\angle ABC = 90°$ 而 $BC = 8$。$D$ 和 $E$ 分別是 $AB$ 和 $AC$ 上的點，使得 $CD = DE$ 及 $\angle DCB = \angle EDA$。若 $\Delta EDC$ 的面積為 50，求 $AB$ 的長度。

   In $\Delta ABC$, $\angle ABC = 90°$ and $BC = 8$. $D$ and $E$ are points on $AB$ and $AC$ respectively such that $CD = DE$ and $\angle DCB = \angle EDA$. If the area of $\Delta EDC$ is 50, find the length of $AB$.




10. 設 $P(x)$ 為四次多項式。若 $P(-1) = -1$、$P(0) = 0$、$P(1) = 1$、$P(2) = 32$ 而 $P(3) = 243$，求 $P(4)$ 的值。

    Let $P(x)$ be a quartic polynomial. If $P(-1) = -1$, $P(0) = 0$, $P(1) = 1$, $P(2) = 32$ and $P(3) = 243$, find the value of $P(4)$.

11. 已知 $\frac{1}{71}$ 以小數表示時，小數點後首 35 個位組成循環節。求此循環節的最後三個位（即從左至右列出小數點後第 33、34 和 35 個位）。

    It is known that when $\frac{1}{71}$ is written as a decimal, the first 35 digits after the decimal point form a recurring period. Find the last three digits of this recurring period, i.e. list the 33rd, 34th and 35th digits after the decimal point from left to right.

12. 已知 $\log 2 \approx 0.30103$ 及 $\log 3 \approx 0.47712$。若 $19^{100}$ 是個 $n$ 位數，求 $n$ 的值。

    Given $\log 2 \approx 0.30103$ and $\log 3 \approx 0.47712$. If $19^{100}$ is an $n$-digit number, find the value of $n$.

**第 13 至第 16 題，每題 6 分。**

**Questions 13 to 16 each carries 6 marks.**

13. 某次測驗共有 6 道選擇題，首 3 題各有 3 個選擇，後 3 題各有 4 個選擇，而每題只有一個正確答案。若可怡每題隨機填上一個選擇，那麼她恰好答對 3 道題目的概率是多少？

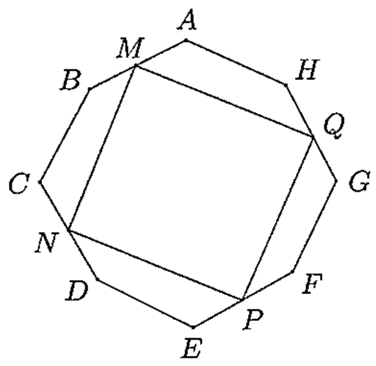
In a test there are 6 multiple choice questions. Each of the first 3 questions has 3 choices, while each of the last 3 questions has 4 choices. There is only one correct answer for each question. If Lydia randomly selects a choice for each question, what is the probability that she gets exactly 3 questions correct?

14. 若 $10\sin\theta^\circ+\cos 2\theta^\circ=2\sin 3\theta^\circ+2$，其中 $0<\theta<90$，求 $\tan\theta^\circ$ 的值。

If $10\sin\theta^\circ+\cos 2\theta^\circ=2\sin 3\theta^\circ+2$ where $0<\theta<90$, find the value of $\tan\theta^\circ$.

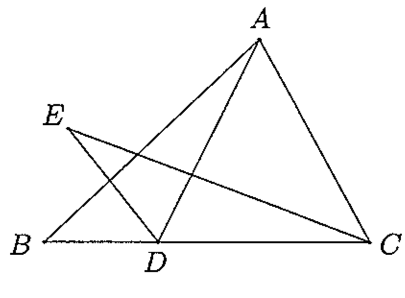
15. 圖中，*ABCDEFGH* 是正八邊形，*M*、*N*、*P*、*Q* 分別是 *AB*、*CD*、*EF* 和 *GH* 的中點。若 *MNPQ* 的面積爲 1，求 *ACEG* 的面積，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 $a$、$b$ 爲有理數。

In the figure, *ABCDEFGH* a regular octagon. *M*, *N*, *P*, *Q* are the mid-points of *AB*, *CD*, *EF* and *GH* respectively. If the area of *MNPQ* is 1, find the area of *ACEG*, giving your answer in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where $a$ and $b$ are rational.




16. 如圖所示，$D$ 是 $\Delta ABC$ 邊 $BC$ 上的一點。已知 $AD=AC=6$、$BD=3$ 及 $CD=5$。若 $E$ 爲點 $D$ 沿線段 $AB$ 反射後所得的點，求 $CE$ 的長度。

As shown in the figure, $D$ is a point on side $BC$ of $\Delta ABC$. Suppose $AD=AC=6$, $BD=3$ and $CD=5$. If $E$ is the point obtained by reflecting point $D$ across line segment $AB$, find the length of $CE$.



**第 17 至第 20 題，每題 7 分。**

**Questions 17 to 20 each carries 7 marks.**

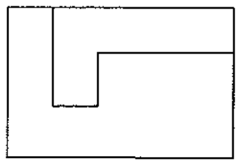
17. 設 $A=(0,1)$ 、 $B=(4,9)$ 和 $C=(x,-1)$ 爲座標平面上的三點。求實數 $x$ 的值，使得 $\angle ACB$ 達至最大值。

    Let $A=(0,1)$, $B=(4,9)$ and $C=(x,-1)$ be three points on the coordinate plane. Find the value of the real number $x$ for which $\angle ACB$ attains its maximum value.

18. 設 $a$、$b$、$c$、$d$、$e$ 爲整數，其中 $(b, c)$ 和 $(d, e)$ 爲拋物線 $y=ax^2+x$ 上的兩點。若這兩點的距離爲 99，求 $a^2-|b^2-d^2|$ 的值。

    Let $a$, $b$, $c$, $d$, $e$ be integers such that $(b, c)$ and $(d, e)$ are two points on the parabola $y=ax^2+x$. If the distance between these two points is 99, find the value of $a^2-|b^2-d^2|$.

19. 一個長方形被分割成一個六邊形和一個八邊形（如圖），分割線均與長方形的邊平行。已知八邊形邊的長度分別爲 1、2、3、4、5、6、7和 8，其中次序並不確定。求六邊形面積的最大可能值。

    A rectangle is divided into a hexagon and an octagon (as shown) such that the cuttings are parallel to the sides of the rectangle. Suppose the side lengths of the octagon are 1, 2, 3, 4, 5, 6, 7 and 8 respectively in some order. Find the largest possible value of the area of the hexagon.

20. 若 $a$、$b$ 爲不同的非負整數，使得 $(6030+a)(6030+b)$ 是平方數，求 $a+b$ 的最小可能值。

    Let $a$ and $b$ be different non-negative integers such that $(6030+a)(6030+b)$ is a square number. Find the smallest possible value of $a+b$.

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**END OF PAPER**