第十三屆培正數學邀請賽（2014 年）

13th Pui Ching Invitational Mathematics Competition (2014)

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確値表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 一個大正方體由 $15\times15\times15$ 個單位正方體組成。有多少個單位正方體的至少一面爲大正方體的表面？

   A larger cube is composed of $15\times15\times15$ unit cubes. How many unit cubes are there with at least one face being the surface of the larger cube?

2. 設 n 爲整數，使得 $\frac{38n-7}{20}<2n+1<\frac{24n-5}{13}$。求 n 所有可能值之和。

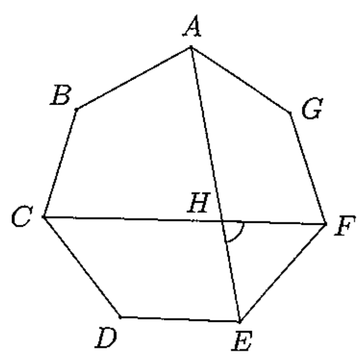
   Let n be an integer such that $\frac{38n-7}{20}<2n+1<\frac{24n-5}{13}$. Find the sum of all possible values of n.

3. 已知方程 $x^3-8x^2-8x-9=0$ 的解是 $x=9$。求方程 $x^6-8x^4-8x^2-9=0$ 的一個正數解。

   Given that the equation $x^3-8x^2-8x-9=0$ has solution $x=9$, find a positive solution to the equation $x^6-8x^4-8x^2-9=0$.

4. 設 $ABCDEFG$ 爲正七邊形，H 爲 AE 與 CF 的交點。若 $\angle FHE=x°$，求 x 的值。

   Let $ABCDEFG$ be a regular heptagon, and H be the intersection of AE and CF. If $\angle FHE=x°$, find the value of x.

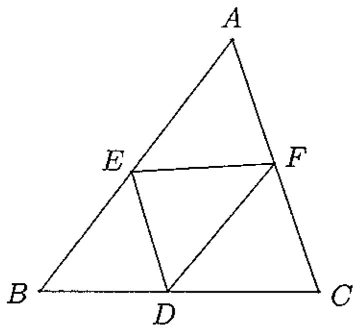


第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

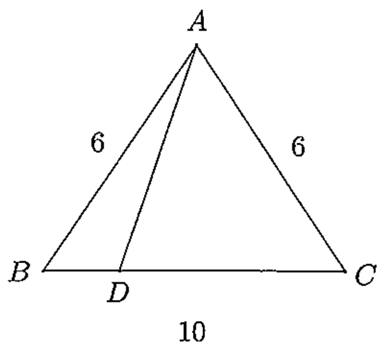
5. 給定 ΔABC，D 是 BC 上的一點，滿足 $BD:DC=3:5$；E 和 F 則分別是 AB 和 AC 上的點，使得 ΔEBD 和 ΔFDC 均與 ΔABC 相似。若 ΔABC 的面積爲 128，求 ΔDEF 的面積。

   Given ΔABC, D is a point on BC satisfying $BD:DC=3:5$. E and F are points on AB and AC respectively, such that both ΔEBD and ΔFDC are similar to ΔABC. If the area of ΔABC is 128, find the area of ΔDEF.



6. 圖中，ABC 是等腰三角形，其中 $AB=AC=6$ 而 $BC=10$。D 是 BC 上的一點，使得 $CA=CD$。求 AD 的長度。

   In the figure, ΔABC is isosceles with $AB=AC=6$ and $BC=10$. D is a point on BC such that $CA=CD$. Find the length of AD.

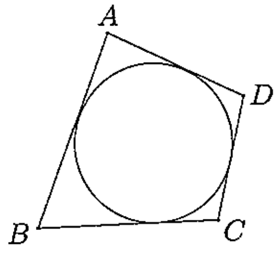


7. 設 $P(x)=x^3+ax^2+bx+c$，其中 a、b、c 爲正整數。已知對任意正整數 m，$P(m)$ 均爲 n 的倍數。求 n 的最大可能值。

   Let $P(x)=x^3+ax^2+bx+c$, where a, b, c are positive integers. It is given that for any positive integer m, $P(m)$ is a multiple of n. Find the greatest possible value of n.

8. 一個圓形內接於四邊形 $ABCD$，其中 $ABCD$ 的周界和面積分別爲 600 和 2014。求該圓形的半徑。

   A circle is inscribed in a quadrilateral $ABCD$, where the perimeter and area of $ABCD$ are 600 and 2014 respectively. Find the radius of the circle.

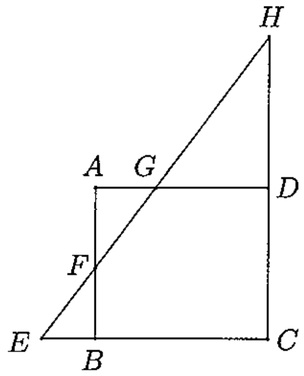


第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

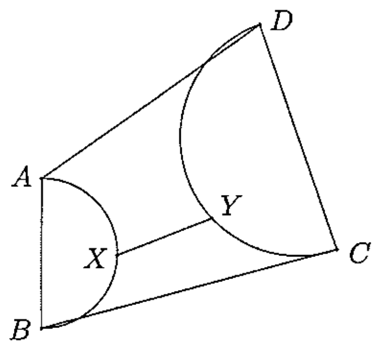
9. 圖中，$ABCD$ 是邊長 24 的正方形，E 和 H 分別是 CB 和 CD 延線上的點，F 和 G 則分別是 EH 與 AB 和 AD 的交點。若 $EF:FG:GH=3:5:7$，求 ΔECH 的面積。

   In the figure, $ABCD$ is a square with side length 24. E and H are points on the extensions of CB and CD respectively, while F and G are the intersections of EH with AB and AD respectively. If $EF:FG:GH=3:5:7$, find the area of ΔECH.



10. 四邊形 $ABCD$ 的頂點的座標分別爲 $A=(0,6)$、$B=(0,0)$、$C=(14,3)$ 和 $D=(10,13)$。以 AB 和 CD 爲直徑向四邊形內作半圓形。若 X 和 Y 分別在兩個半圓形上，求 X 和 Y 之間距離的最小可能值。

    The coordinates of the vertices of a quadrilateral $ABCD$ are $A=(0,6)$, $B=(0,0)$, $C=(14,3)$ and $D=(10,13)$ respectively. Construct semi-circles towards the interior of the quadrilateral with diameters AB and CD. If X and Y are points on the two semi-circles respectively, find the smallest possible value of the distance between X and Y.



11. 小莉和小美進行乒乓球比賽，先取得 7 局勝利的一方爲優勝者。已知在首 6 局中，小莉勝出 4 局而小美勝出 2 局。若每局比賽中兩人獲勝的概率相同，且沒有平局，求小莉最後勝出的概率。

    Lily and Mimi are having a table tennis competition. The one who wins 7 games first will become the winner. It is known that during the first 6 games, Lily has won 4 while Mimi has won 2. If they have the same chance of winning each game and there are no ties, find the probability for Lily to be the winner in the end.

12. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求乘積 TITBIT 所代表的六位數。

    In the multiplication shown, each letter represents a different digit from 0 to 9. Find the six-digit number represented by the product TITBIT.

$$\begin{array}{r} \text{C E D E D} \\ \times \quad \text{4 2} \\ \hline \text{T I T B I T} \end{array}$$

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 某次測驗共有 6 道選擇題，首 3 題各有 3 個選擇，後 3 題各有 4 個選擇，而每題只有一個正確答案。若可怡每題隨機填上一個選擇，那麼她恰好答對 3 道題目的概率是多少？

    In a test there are 6 multiple choice questions. Each of the first 3 questions has 3 choices, while each of the last 3 questions has 4 choices. There is only one correct answer for each question. If Lydia randomly selects a choice for each question, what is the probability that she gets exactly 3 questions correct?

14. 已知 n 是兩位正整數，其數字之積是其數字之和的倍數。問 n 有多少個不同的可能值？

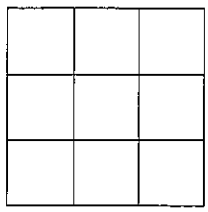
    Given n is a two-digit positive integer whose product of digits is a multiple of its sum of digits. How many different possible values of n are there?

15. 若 m、n 爲正整數，且 $m^2-3n^2=9477$，求 m 的最小可能值。

    If m and n are positive integers such that $m^2-3n^2=9477$, find the smallest possible value of m.

16. 有一個 3×3 的方格表，現要把當中一個或以上的方格塗黑，但規定兩個有公共邊的方格不能同時塗黑，那麼共有多少種不同的塗色方法？（通過旋轉一種塗色方法而得到的新塗色方法視爲跟原方法相同。）

    There is a 3×3 grid and one or more cells are to be coloured black, subject to the requirement that no two cells having a common edge can be simultaneously coloured black. How many different colourings are there? (A colouring which is obtained by rotating an existing colouring is regarded to be the same as the original colouring.)



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 一隻草蜢從某地出發，每次牠可以向東或向西跳 1 個單位。該草蜢有多少種方法可以跳 14 次後返回原位，而且途中不曾經過原位？

A grasshopper starts at a certain place and jumps 1 unit to the east or west each time. How many ways are there for the grasshopper to return to the starting position in 14 jumps such that it does not pass through the starting position during the intermediate steps?

18. 設 p、q 和 r 爲互不相同的質數， a、b、c 爲滿足 $a>b>c\geq 0$ 的整數。若 $p(q+1)+q(r+1)+r(p+1)=a^2+b^2+c^2-pqr$ ，求 a 的最小可能值。

Let p, q, r be pairwise distinct prime numbers, and a, b, c be integers satisfying $a>b>c\geq 0$. If $p(q+1)+q(r+1)+r(p+1)=a^2+b^2+c^2-pqr$, find the smallest possible value of a.

19. 已知在 m 進制（其中 $1<m\leq 100$ ）中，存在數字 A、B 使得 $\sqrt{\overline{AB}}=A\sqrt{B}$ 成立（這裡 $\overline{AB}$ 代表 m 進制中的一個兩位數）。求 m 的所有可能值之和。

It is given that in base m notation (where $1<m\leq 100$), there exist digits A, B for which $\sqrt{\overline{AB}}=A\sqrt{B}$ (here $\overline{AB}$ denotes a two-digit number in base m notation). Find the sum of all possible values of m.

20. 現要把 1 至 9 這九個數字填進下式的九個空格中使等式成立：

$$\square\square\times(\square\square+\square)-\square\square\div\square\square=2014$$

求可以填進首兩格的兩位數的所有可能值之和。

The nine digits 1 to 9 are to be filled into the nine boxes below to make equality hold:

$$\square\square\times(\square\square+\square)-\square\square\div\square\square=2014$$

What is the sum of all possible two-digit numbers that can be filled into the first two boxes?

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END OF PAPER