第十三屆培正數學邀請賽（2014 年）

13th Pui Ching Invitational Mathematics Competition (2014)

決賽（中二組）

Final Event (Secondary 2)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 在大於 1234 而小於 2014 的整數當中，有多少個是 3 的倍數？

   How many of the integers greater than 1234 and smaller than 2014 are multiples of 3?

2. 有多少個四位正整數的千位和百位相同，且十位和個位相同？

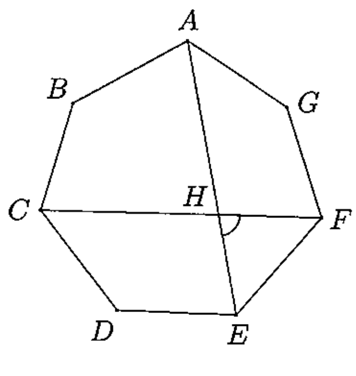
   How many four-digit positive integers are there such that the thousands and hundreds digits are equal, and the tens and unit digits are equal?

3. 設 a_1、a_2、…、a_n 爲正整數，且 $a_1!+a_2!+\cdots+a_n!=2014$（其中 $m!=1\times 2\times\cdots\times m$）。求 n 的最小可能值。

   Let a_1, a_2, …, a_n be positive integers such that $a_1!+a_2!+\cdots+a_n!=2014$ (where $m!=1\times 2\times\cdots\times m$). Find the smallest possible value of n.

4. 設 $ABCDEFG$ 爲正七邊形，H 爲 AE 與 CF 的交點。若 $\angle FHE = x^\circ$，求 x 的值。

   Let $ABCDEFG$ be a regular heptagon, and H be the intersection of AE and CF. If $\angle FHE = x^\circ$, find the value of x.



第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 某些正整數之和爲 19，積爲 n。求 n 的最大可能值。

   The sum of several positive integers is 19 and their product is n. Find the greatest possible value of n.

6. 若 $a+\frac{3}{a}=5$，求 $\frac{(a+2)^2-1}{a^3(5a-22)}$ 的值。

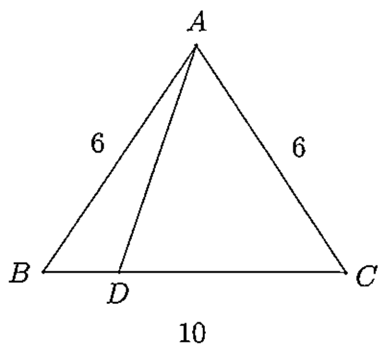
   If $a+\frac{3}{a}=5$, find the value of $\frac{(a+2)^2-1}{a^3(5a-22)}$.

7. 設 m、n 爲正整數，使得 $m^3+2n^2=801$。若 m 和 n 的最大公因數是 d，求 d 的所有可能值之和。

   Suppose m and n are positive integers satisfying $m^3+2n^2=801$. If the H.C.F. of m and n is d, find the sum of all possible values of d.

8. 圖中，ABC 是等腰三角形，其中 $AB=AC=6$ 而 $BC=10$。D 是 BC 上的一點，使得 $CA=CD$。求 AD 的長度。

   In the figure, ΔABC is isosceles with $AB=AC=6$ and $BC=10$. D is a point on BC such that $CA=CD$. Find the length of AD.

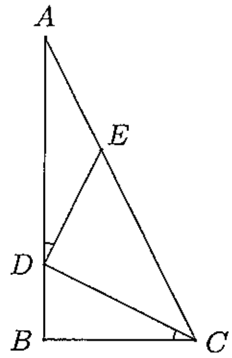


第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

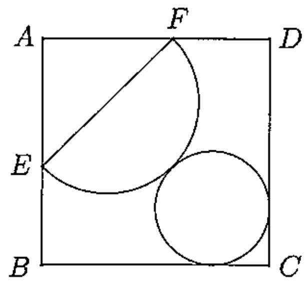
9. 在 ΔABC 中，$\angle ABC = 90°$ 而 $BC = 8$。D 和 E 分別是 AB 和 AC 上的點，使得 $CD = DE$ 及 $\angle DCB = \angle EDA$。若 ΔEDC 的面積爲 50，求 AB 的長度。

   In ΔABC, $\angle ABC = 90°$ and $BC = 8$. D and E are points on AB and AC respectively such that $CD = DE$ and $\angle DCB = \angle EDA$. If the area of ΔEDC is 50, find the length of AB.



10. $ABCD$ 是邊長爲 2 的正方形。E 和 F 分別是 AB 和 AD 的中點。以 EF 爲直徑作半圓。若存在另一圓形與半圓和邊 CB 及 CD 相切，求此圓形的半徑，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 a、b 爲有理數。

    Given a square $ABCD$ with side length 2. Let E and F be the midpoints of sides AB and AD respectively. Construct a semi-circle with EF as the diameter. If there exists another circle which is tangent to the semi-circle and to sides CB and CD, find the radius of this circle, giving your answer in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where a and b are rational.



11. 已知 $\frac{1}{71}$ 以小數表示時，小數點後首 35 個位組成循環節。求此循環節的最後三個位（即從左至右列出小數點後第 33、34 和 35 個位）。

    It is known that when $\frac{1}{71}$ is written as a decimal, the first 35 digits after the decimal point form a recurring period. Find the last three digits of this recurring period, i.e. list the 33rd, 34th and 35th digits after the decimal point from left to right.

12. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求乘積 TITBIT 所代表的六位數。

    In the multiplication shown, each letter represents a different digit from 0 to 9. Find the six-digit number represented by the product TITBIT.

$$\begin{array}{r} \text{C E D E D} \\ \times \qquad\quad \text{4 2} \\ \hline \text{T I T B I T} \end{array}$$

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 求一個約等於 0.001001004007019040（準確至小數點後 18 個位）且分母小於 1000000 的分數。（提示：數列 1, 1, 4, 7, 19, 40, … 滿足關係式 $a_{n+2}=a_{n+1}+3a_n$。）

    Find a fraction, with denominator less than 1000000, which is approximately equal to 0.001001004007019040 (correct to 18 decimal places). (Hint: The sequence 1, 1, 4, 7, 19, 40, … satisfies the recurrence relation $a_{n+2}=a_{n+1}+3a_n$.)

14. 設 $a_1=2014$，並對正整數 n 定義 $a_{n+1}=a_1+a_2+\cdots+a_n+n$。求 a_{2014} 的最後兩位數字。

    Let $a_1=2014$ and define $a_{n+1}=a_1+a_2+\cdots+a_n+n$ for positive integers n. Find the last two digits of a_{2014}.

15. 已知 n 是兩位正整數，其數字之積是其數字之和的倍數。問 n 有多少個不同的可能值？

    Given n is a two-digit positive integer whose product of digits is a multiple of its sum of digits. How many different possible values of n are there?

16. 有些五位正整數的五個數字互不相同。在這些整數中，可被 4 整除的佔幾分之幾？

    Some five-digit positive integers consist of five different digits. Among these integers, what fraction of them are divisible by 4?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 一隻草蜢從某地出發，每次牠可以向東或向西跳 1 個單位。該草蜢有多少種方法可以跳 14 次後返回原位，而且途中不曾經過原位？

    A grasshopper starts at a certain place and jumps 1 unit to the east or west each time. How many ways are there for the grasshopper to return to the starting position in 14 jumps such that it does not pass through the starting position during the intermediate steps?

18. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1]=1$、$[6.9]=6$ 和 $[5]=5$；y^+ 表示 y 和 0 兩者中較大的一個，例如 $1.1^+=1.1$ 和 $(-6.9)^+=0$。定義 $a_k=\left[\sqrt{\left(4102-\frac{4102}{2014}k^2\right)^+}\right]$ 和 $b_k=\left[\sqrt{\left(2014-\frac{2014}{4102}k^2\right)^+}\right]$。求 $(a_0+a_1+\cdots+a_{100})-(b_0+b_1+\cdots+b_{100})$ 的值。

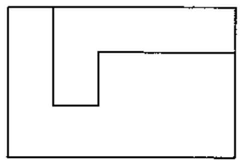
Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1]=1$, $[6.9]=6$ and $[5]=5$. Let also y^+ denote the greater of the two numbers y and 0. For example, $1.1^+=1.1$ and $(-6.9)^+=0$. Define $a_k=\left[\sqrt{\left(4102-\frac{4102}{2014}k^2\right)^+}\right]$ and $b_k=\left[\sqrt{\left(2014-\frac{2014}{4102}k^2\right)^+}\right]$. Find the value of $(a_0+a_1+\cdots+a_{100})-(b_0+b_1+\cdots+b_{100})$.

19. 已知在 m 進制（其中 $1<m\leq 100$）中，存在數字 A、B 使得 $\sqrt{\overline{AB}}=A\sqrt{B}$ 成立（這裡 $\overline{AB}$ 代表 m 進制中的一個兩位數）。求 m 的所有可能值之和。

It is given that in base m notation (where $1<m\leq 100$), there exist digits A, B for which $\sqrt{\overline{AB}}=A\sqrt{B}$ (here $\overline{AB}$ denotes a two-digit number in base m notation). Find the sum of all possible values of m.

20. 一個長方形被分割成一個六邊形和一個八邊形（如圖），分割線均與長方形的邊平行。已知八邊形邊的長度分別爲 1、2、3、4、5、6、7 和 8，其中次序並不確定。求六邊形面積的最大可能值。

A rectangle is divided into a hexagon and an octagon (as shown) such that the cuttings are parallel to the sides of the rectangle. Suppose the side lengths of the octagon are 1, 2, 3, 4, 5, 6, 7 and 8 respectively in some order. Find the largest possible value of the area of the hexagon.



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