第十三屆培正數學邀請賽（2014年）

13th Pui Ching Invitational Mathematics Competition (2014)

決賽（中一組）

Final Event (Secondary 1)

時限：2小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設20題，總分爲100分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 一個大正方體由 $15\times15\times15$ 個單位正方體組成。有多少個單位正方體的至少一面爲大正方體的表面？

   A larger cube is composed of $15\times15\times15$ unit cubes. How many unit cubes are there with at least one face being the surface of the larger cube?

2. 在大於 1234 而小於 2014 的整數當中，有多少個是 3 的倍數？

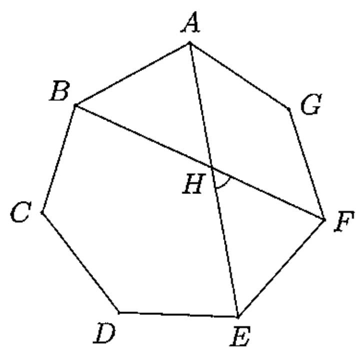
   How many of the integers greater than 1234 and smaller than 2014 are multiples of 3?

3. 已知方程 $x^3-8x^2-8x-9=0$ 的解是 $x=9$。求方程 $x^6-8x^4-8x^2-9=0$ 的一個正數解。

   Given that the equation $x^3-8x^2-8x-9=0$ has solution $x=9$, find a positive solution to the equation $x^6-8x^4-8x^2-9=0$.

4. 設 *ABCDEFG* 爲正七邊形，*H* 爲 *AE* 與 *BF* 的交點。若 $\angle FHE = x°$，求 x 的值。

   Let *ABCDEFG* be a regular heptagon, and *H* be the intersection of *AE* and *BF*. If $\angle FHE = x°$, find the value of x.



第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 某些正整數之和爲 19，積爲 n。求 n 的最大可能值。

   The sum of several positive integers is 19 and their product is n. Find the greatest possible value of n.

6. 設 m、n 爲正整數，使得 $m^3+2n^2=801$。若 m 和 n 的最大公因數是 d，求 d 的所有可能值之和。

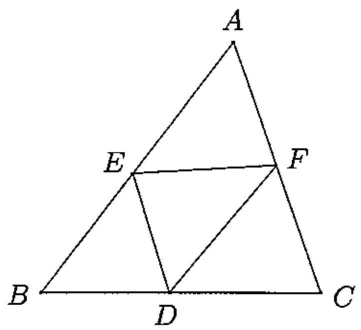
   Suppose m and n are positive integers satisfying $m^3+2n^2=801$. If the H.C.F. of m and n is d, find the sum of all possible values of d.

7. 現有五張分別寫上 0、1、1、2、3 的咭片。若把它們以不同的次序排成一行，共可得到多少個不同的五位正奇數？

   There are five cardboards with 0, 1, 1, 2, 3 written respectively. By arranging them in a row in different orders, how many different five-digit positive odd numbers can be formed?

8. 給定 ΔABC，D 是 BC 上的一點，滿足 $BD:DC=3:5$；E 和 F 則分別是 AB 和 AC 上的點，使得 ΔEBD 和 ΔFDC 均與 ΔABC 相似。若 ΔABC 的面積爲 128，求 ΔDEF 的面積。

   Given ΔABC, D is a point on BC satisfying $BD:DC=3:5$. E and F are points on AB and AC respectively, such that both ΔEBD and ΔFDC are similar to ΔABC. If the area of ΔABC is 128, find the area of ΔDEF.

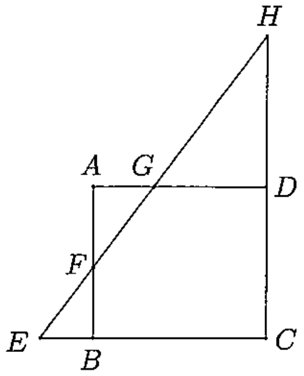


第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 圖中，$ABCD$ 是邊長 24 的正方形，E 和 H 分別是 CB 和 CD 延線上的點，F 和 G 則分別是 EH 與 AB 和 AD 的交點。若 $EF:FG:GH=3:5:7$，求 ΔECH 的面積。

   In the figure, $ABCD$ is a square with side length 24. E and H are points on the extensions of CB and CD respectively, while F and G are the intersections of EH with AB and AD respectively. If $EF:FG:GH=3:5:7$, find the area of ΔECH.



10. 現有一部神奇的機器，每次輸入一個小於 1 的正數時，它會把該數小數點後的第一個數字擦掉，並輸出一個新的數。例如，若輸入 0.346，它會輸出 0.46；若輸入 $0.3\dot{4}\dot{6}$，它會輸出 $0.\dot{4}6\dot{3}$。已知輸入 x 後（其中 $0<x<1$），它會輸出 $3x$。求 x 所有可能值之和。

    There is a magic machine. Each time when a positive number less than 1 is input, it will delete the digit immediately after the decimal point and output a new number. For example, if the input is 0.346, the output will be 0.46; if the input is $0.3\dot{4}\dot{6}$, the output will be $0.\dot{4}6\dot{3}$. It is given that when x is input (where $0<x<1$), the output will be $3x$. Find the sum of all possible values of x.

11. 一隻草蜢位於一個 3×3 的方格表的中央方格中，每次牠可以跳到一個不處於同一行亦不處於同一列的另一個方格。那麼該草蜢有多少種方法可以跳動 4 次，使牠不曾處於同一格子兩次？

    A grasshopper is lying at the centre cell of a 3×3 grid. Each time it can jump to another cell which is neither in the same row nor in the same column as the previous cell. How many ways are there for the grasshopper to jump 4 times so that it never stays in the same cell twice?

12. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字。求乘積 TITBIT 所代表的六位數。

    In the multiplication shown, each letter represents a different digit from 0 to 9. Find the six-digit number represented by the product TITBIT.

$$\begin{array}{r} \text{C E D E D} \\ \times \qquad\quad \text{4 2} \\ \hline \text{T I T B I T} \end{array}$$

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 求一個約等於 0.001001004007019040（準確至小數點後 18 個位）且分母小於 1000000 的分數。（提示：數列 1, 1, 4, 7, 19, 40, ... 滿足關係式 $a_{n+2} = a_{n+1} + 3a_n$ 。）

Find a fraction, with denominator less than 1000000, which is approximately equal to 0.001001004007019040 (correct to 18 decimal places). (Hint: The sequence 1, 1, 4, 7, 19, 40, ... satisfies the recurrence relation $a_{n+2} = a_{n+1} + 3a_n$.)

14. 設 $a_1 = 2014$，並對正整數 n 定義 $a_{n+1} = a_1 + a_2 + \cdots + a_n + n$。求 a_{2014} 的最後兩位數字。

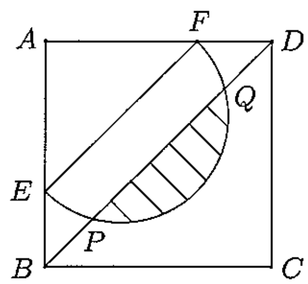
Let $a_1 = 2014$ and define $a_{n+1} = a_1 + a_2 + \cdots + a_n + n$ for positive integers n. Find the last two digits of a_{2014}.

15. 有些五位正整數的五個數字互不相同。在這些整數中，可被 4 整除的佔幾分之幾？

Some five-digit positive integers consist of five different digits. Among these integers, what fraction of them are divisible by 4?

16. $ABCD$ 是邊長爲 3 的正方形，E 和 F 分別是 AB 和 AD 上的點，使得 $AE = AF = 2$。以 EF 爲直徑作半圓，與對角線 BD 相交於 P 和 Q。求圖中陰影部分的面積。

Given a square $ABCD$ with side length 3. Let E and F be points on sides AB and AD respectively, such that $AE = AF = 2$. Construct a semi-circle with EF as the diameter. It intersects the diagonal BD at P and Q. Find the area of the shaded region as shown in the diagram.



第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 p、q 和 r 爲互不相同的質數，a、b、c 爲滿足 $a > b > c \geq 0$ 的整數。若 $p(q+1) + q(r+1) + r(p+1) = a^2 + b^2 + c^2 - pqr$，求 a 的最小可能值。

Let p, q, r be pairwise distinct prime numbers, and a, b, c be integers satisfying $a > b > c \geq 0$. If $p(q+1) + q(r+1) + r(p+1) = a^2 + b^2 + c^2 - pqr$, find the smallest possible value of a.

18. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1]=1$、$[6.9]=6$ 和 $[5]=5$；y^+ 表示 y 和 0 兩者中較大的一個，例如 $1.1^+=1.1$ 和 $(-6.9)^+=0$。定義 $a_k=\left[\sqrt{\left(4102-\frac{4102}{2014}k^2\right)^+}\right]$ 和 $b_k=\left[\sqrt{\left(2014-\frac{2014}{4102}k^2\right)^+}\right]$。求 $(a_0+a_1+\cdots+a_{100})-(b_0+b_1+\cdots+b_{100})$ 的值。

Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1]=1$, $[6.9]=6$ and $[5]=5$. Let also y^+ denote the greater of the two numbers y and 0. For example, $1.1^+=1.1$ and $(-6.9)^+=0$. Define $a_k=\left[\sqrt{\left(4102-\frac{4102}{2014}k^2\right)^+}\right]$ and $b_k=\left[\sqrt{\left(2014-\frac{2014}{4102}k^2\right)^+}\right]$. Find the value of $(a_0+a_1+\cdots+a_{100})-(b_0+b_1+\cdots+b_{100})$.

19. 若 a、b 爲不同的非負整數，使得 $(6030+a)(6030+b)$ 是平方數，求 $a+b$ 的最小可能值。

Let a and b be different non-negative integers such that $(6030+a)(6030+b)$ is a square number. Find the smallest possible value of $a+b$.

20. 現要把 1 至 9 這九個數字填進下式的九個空格中使等式成立：

$$\square\square\times(\square\square+\square)-\square\square\div\square\square=2014$$

求可以填進首兩格的兩位數的所有可能值之和。

The nine digits 1 to 9 are to be filled into the nine boxes below to make equality hold:

$$\square\square\times(\square\square+\square)-\square\square\div\square\square=2014$$

What is the sum of all possible two-digit numbers that can be filled into the first two boxes?

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END OF PAPER