第十二屆培正數學邀請賽（2013 年）

12th Pui Ching Invitational Mathematics Competition (2013)

初賽（中四組）

Heat Event (Secondary 4)

時限：1 小時 15 分

Time allowed: 1 hour 15 minutes

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 作答時，每題的答案均須以 0 至 9999 之間的整數表示。依照答題紙上的指示填寫答案，毋須呈交計算步驟。

Each answer must be given in the form of an integer between 0 and 9999. Follow the instructions on the answer sheet to enter the answers. You are not required to hand in your steps of working.

(d) 不得使用計算機。

The use of calculators is not allowed.

(e) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：每題的答案均須以 0 至 9999 之間的整數表示，如有需要應以上述範圍內最接近正確答案的整數回答。如有兩個這樣的整數與正確答案同樣接近，則以「四捨五入」的原則取較大的整數。請細閱答題紙上的指示。

Note: Each answer must be given in the form of an integer between 0 and 9999. Where necessary, the answer should be rounded off to the nearest integer in the above range. Read the instructions on the answer sheet in detail.

1. 若 x、y、z 爲正整數，使得 $x^2+y^2=xyz$，求 z。（3 分）

   If x, y, z are positive integers such that $x^2+y^2=xyz$, find z. (3 marks)

2. 求 3^{33} 除以 121 時的餘數。（3 分）

   Find the remainder when 3^{33} is divided by 121. (3 marks)

3. 把 $y=3\sqrt[4]{x}$ 的圖像沿直線 $y=x$ 反射可得 $y=ax^b$（其中 a、b 爲常數）的圖像。求 $\frac{b}{a}$ 的值。（4 分）

   By reflecting the graph of $y=3\sqrt[4]{x}$ along the line $y=x$, one obtains the graph of $y=ax^b$ where a and b are constants. Find the value of $\frac{b}{a}$. (4 marks)

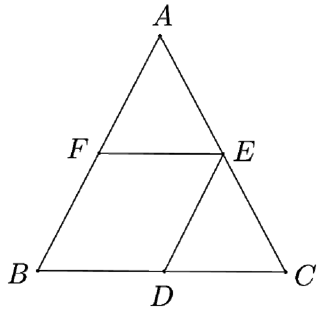
4. 某學校每名學生的編號均由四個數字組成（容許以 0 開首），且當中最少一個數字是 1 或 2。已知沒有兩名學生的編號相同，則該校最多有幾名學生？（4 分）

   In a school, each student has a number consisting of four digits (which could begin with 0) in which at least one digit is 1 or 2. Given that no two students have the same number, what is the maximum number of students in the school? (4 marks)

5. 有多少種方法可以從一個正十二邊形的頂點中選取三點，使得它們組成一個銳角三角形？（4 分）

   How many ways are there to choose three points from the vertices of a regular 12-sided polygon such that they can form an acute-angled triangle? (4 marks)

6. 圖中，D、E、F 分別爲 BC、CA 和 AB 上的點，使得 $BDEF$ 爲平行四邊形。若 ΔAFE 和 $BDEF$ 的面積均爲 100，求 ΔABC 的面積。（4分）



In the figure, D, E, F are points on BC, CA and AB respectively such that $BDEF$ form a parallelogram. It is known that the area of ΔAFE and $BDEF$ are both 100. Find the area of ΔABC. (4 marks)

7. 已知正整數 n 爲 2013 的倍數，求 n 的數字之和的最小可能值。（4分）

Given that the positive integer n is a multiple of 2013. Find the smallest possible value of the sum of digits of n. (4 marks)

8. 若 θ 滿足 $0<\theta<90$ 及 $(3\sin\theta^\circ+\cos\theta^\circ)\sin\theta^\circ=2$ ，求 θ 的值。（4分）

If θ satisfies $0<\theta<90$ and $(3\sin\theta^\circ+\cos\theta^\circ)\sin\theta^\circ=2$, find the value of θ. (4 marks)

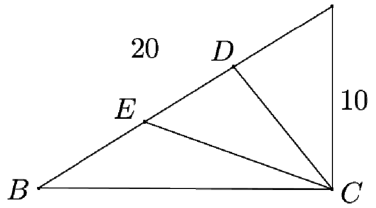
9. 已知方程 $x^3+ax^2-17x+1=0$ （其中 a 是整數）有有理數解，求 a 的最大可能值。（4分）

If the equation $x^3+ax^2-17x+1=0$ (where a is an integer) has rational solution, find the greatest possible value of a. (4 marks)

10. 老師有紅色、黃色和藍色的糖果各兩顆，並打算給小陳、小李、小張、小黃和小何五名學生每人一顆，問老師有多少種分配糖果的方法？（5分）

The teacher has two red, two yellow and two blue candies, and plans to give one candy to each of Ann, Ben, Carl, Dan and Eva. In how many different ways can the teacher distribute the candies? (5 marks)

11. 在 ΔABC 中，$\angle C=90^\circ$、$AB=20$、$AC=10$。設 D、E 爲 AB 上的點，使得 $\angle ACD=\angle DCE=\angle ECB$。求 ΔCDE 的周界。（5分）



In ΔABC, $\angle C=90^\circ$, $AB=20$ and $AC=10$. D and E are points on AB such that $\angle ACD=\angle DCE=\angle ECB$. Find the perimeter of ΔCDE. (5 marks)

12. 在所示的算式中，每個字母代表一個由 1 至 9 的不同數字。求 A＋B＋C＋D＋E 的值。（5 分）

In the addition shown, each letter represents a different digit from 1 to 9. Find the value of A＋B＋C＋D＋E. (5 marks)

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    A A B
    C A C
+   C B B
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  D E 0 0
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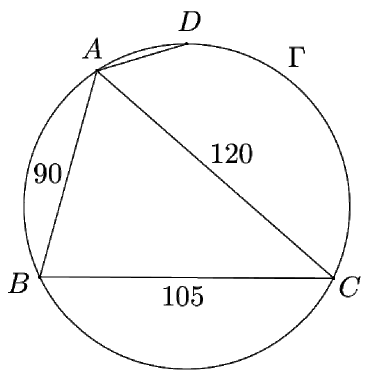
13. 已知方程 $x^3-9x^2+15x+c=0$（其中 c 爲常數）有一重根，求該重根的所有可能值之和。（5 分）

It is known that the equation $x^3-9x^2+15x+c=0$ (where c is a constant) has a multiple root. Find the sum of all possible values of that multiple root. (5 marks)

14. 如圖所示，Γ 爲 ΔABC 的外接圓，D 爲大弧 $\overset{\frown}{BC}$ 的中點，使得 A、D 在直線 BC 的同側。若 $AB=90$、$BC=105$ 而 $CA=120$，求 AD 的長度。（5 分）

In the figure, Γ is the circumcircle of ΔABC. D is the midpoint of the major arc $\overset{\frown}{BC}$ such that A and D lie on the same side of BC. If $AB=90$, $BC=105$ and $CA=120$, find the length of AD. (5 marks)



15. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1]=1$、$[6.9]=6$ 和 $[5]=5$。求 $\sum_{i=1}^{10}\sum_{j=0}^{i-1}\left[\pi+\frac{j}{i}\right]$ 的值。（6 分）

Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1]=1$, $[6.9]=6$ and $[5]=5$. Evaluate $\sum_{i=1}^{10}\sum_{j=0}^{i-1}\left[\pi+\frac{j}{i}\right]$. (6 marks)

16. 求方程 $\sqrt{x-1}+\sqrt[4]{13-x}=2\sqrt{3}$ 的所有實根之和。（7 分）

Find the sum of all real roots to the equation $\sqrt{x-1}+\sqrt[4]{13-x}=2\sqrt{3}$. (7 marks)

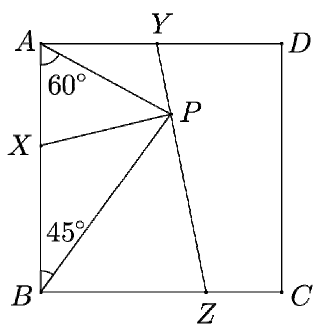
17. 有多少種方法從 1 至 30 中（包括首尾兩數）選出兩個整數 x 和 y（其中 $x>y$），使得 $x+y$ 和 $x-y$ 的最大公因數是 1？（7 分）

In how many different ways can two integers x and y (where $x>y$) be chosen from 1 to 30 (inclusive) such that the H.C.F. of $x+y$ and $x-y$ is 1? (7 marks)

18. 老師跟可晴和芷瑜二人玩遊戲，老師說出一個正整數，然後二人各自把數寫在黑板上。可晴每次擦掉整數最左方的一個數字，而芷瑜每次則擦掉整數最右方的一個數字，直至只剩下一個一位數爲止。每次擦掉一個數字後，如果剩下的整數是偶數，她們便可得到糖果，所得的數目是該數的質因數連乘式中 2 的次數（例如，如果剩下的整數是 1800，則因爲 $1800 = 2^3 \times 3^2 \times 5^2$，所以可得 3 顆糖果）。舉例說，如果老師說出的正整數是 2013，則過程中可晴所得的整數分別是 013、13 和 3，由於全是奇數，故不獲任何糖果；芷瑜所得的整數則分別是 201、20 和 2，在 20 和 2 時分別可獲 2 顆和 1 顆糖果。如果老師開始時說出的正整數是一個等於 2^{119} 的 36 位數，求二人所得糖果數目之差。 （7 分）

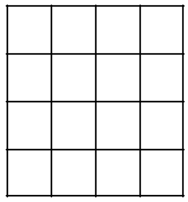
The teacher played a game with Crystal and Rose, as follows. The teacher announced a positive integer, which each of Crystal and Rose wrote on the board. Each time Crystal deletes the leftmost digit of the integer while Rose deletes the rightmost digit, until a one-digit number remains. Each time when a digit is removed, candies will be given if the remaining integer is even. The number of candies given is the power of 2 in the prime factorisation of that number. (For instance, if the number that remains is 1800 then 3 candies will be given since $1800 = 2^3 \times 3^2 \times 5^2$.) As an example, if the teacher announced 2013, then Crystal would obtain 013, 13 and 3 in the process, and will not get any candy since all these numbers are odd; on the other hand Rose would obtain 201, 20 and 2, and would get 2 and 1 candies at 20 and 2 respectively. If the teacher began by announcing a 36-digit number which is equal to 2^{119}, find the difference between the number of candies received by Crystal and Rose. (7 marks)

19. 如圖所示，$ABCD$ 是正方形，邊長爲 30。P 是正方形內的一點，使得 $\angle PAB = 60°$ 和 $\angle PBA = 45°$。X、Y 和 Z 分別是 AB、AD 和 BC 上的點，使得 YZ 穿過 P 且垂直於 PX。求 ΔXYZ 的面積的最大可能值。 （7 分）



$ABCD$ is a square with side length 30. P is a point inside the square such that $\angle PAB = 60°$ and $\angle PBA = 45°$. Let X, Y, Z be points on AB, AD and BC respectively such that YZ passes through point P and is perpendicular to PX. Find the maximum possible area of ΔXYZ. (7 marks)

20. 40 支火柴如圖所示被排列成一個 4×4 的正方形。至少需從中取走多少支火柴，才可令圖中不出現任何大小的正方形？（7 分）



40 matches are arranged to form a 4×4 square as shown. At least how many matches must be removed so that no square of any size remains in the diagram? (7 marks)

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