第十二屆培正數學邀請賽（2013年）

12th Pui Ching Invitational Mathematics Competition (2013)

決賽（高中組）

Final Event (Senior Secondary)

時限：2小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設20題，總分爲100分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 求滿足 $(\log_{10} x)(\log_{10} x^2 + 1) = 1$ 的 x 的最大值。

   Find the largest possible value of x which satisfies $(\log_{10} x)(\log_{10} x^2 + 1) = 1$.

2. 當 345345347349 除以 91 時，餘數是多少？

   What is the remainder when 345345347349 is divided by 91?

3. 當 2013 除以某個兩位正整數時，商和餘數是相同的兩位數。求商。

   When 2013 is divided by a two-digit positive integer, the quotient and remainder are the same two-digit number. Find the quotient.

4. 黑板上寫著 1 至 2013 這 2013 正整數。若把 1 至 9 這九個數字按其在黑板上出現的次數排列成一個九位數（出現次數較少的排在左邊，如果次數相同則較小的數字排在左邊），那麼所得的九位數是甚麼？

   On the blackboard the 2013 positive integers from 1 to 2013 are written. If the nine digits 1 to 9 are rearranged to form a nine-digit number (those occurring less frequently on the left, and in case of equal frequency the smaller digit on the left), what is the nine-digit number formed?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 對於正整數 n，設 $f(n)$ 表示 n 的數字之和，例如：因爲 2013 的數字之和是 6，所以 $f(2013)=6$。若 m 是小於 2013 的正整數，且 $[f(m)]^6<m$，問 m 有多少個不同的可能值？

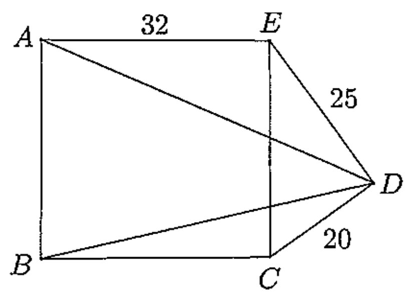
   For positive integer n, let $f(n)$ denote the sum of the digits of n. For example, since the sum of the digits of 2013 is 6, we have $f(2013)=6$. If m is a positive integer less than 2013 and $[f(m)]^6<m$, how many different possible values of m are there?

6. 若長方形 $ABCD$ 的周界爲 98，而其對角線長度爲 $\sqrt{2013}$，求 $ABCD$ 的面積。

   If the perimeter of rectangle $ABCD$ is 98 and the length of its diagonal is $\sqrt{2013}$, find the area of $ABCD$.

7. 圖中，$ABCDE$ 是凸五邊形，$ABCE$ 是一個邊長爲 32 的正方形，且 $CD=20$、$DE=25$。求 AD^2-BD^2 的值。

   In the figure, $ABCDE$ is a convex pentagon while $ABCE$ is a square with side length 32. Furthermore, $CD=20$ and $DE=25$. Find the value of AD^2-BD^2.



8. 某數學比賽有 20 道題，當中 3、4、5、6、7 分題各有 4 題，合共 100 分。每題答對可得該題所佔的分數，否則該題得 0 分。若百楠在比賽中答對了 17 題並獲得 n 分，求 n 所有可能值之和。

   In a mathematical competition there are 20 problems. There are different types of problems carrying 3, 4, 5, 6, 7 marks respectively, with 4 problems for each type, making a total of 100 marks. A correct answer merits all the marks allocated to the problem; otherwise 0 mark is given to the problem. If Jeffrey answers 17 questions correctly in the competition and gets n marks, find the sum of all possible values of n.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 在所示 n 進制的算式中，每個字母代表一個由 1 至 $(n-1)$ 的不同數字。求 n 的值。

In the addition in base n as shown, each letter represents a different digit from 1 to $(n-1)$. Find the value of n.

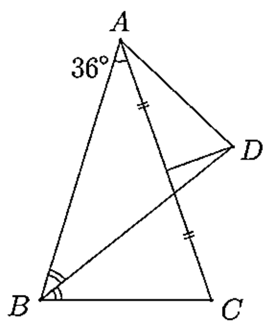
$$\begin{array}{cccccc} & & \mathrm{A} & \mathrm{B} & \mathrm{A} & \mathrm{B} \\ + & & 4 & 0 & 4 & 4 \\ \hline & 1 & 0 & 0 & 0 & \mathrm{A} \end{array}$$

10. 在一個遊戲中，每名參加者均獲發一張遊戲咭，上面印有 1 至 8 這 8 個數字。參加者需要圈出遊戲咭上的其中 3 個數字。然後兩位特別嘉賓會各自獲發一張遊戲咭，他們每人均會隨意圈出其中 3 個數字。如果某位參加者圈出的數字中，最少有一個數字同時被其中一位或兩位特別嘉賓圈出，該參加者便可獲獎。那麼，一名參加者能在遊戲中獲獎的概率是多少？

In a game, each participant is given a card on which the numbers 1 to 8 are printed. A participant has to circle 3 numbers on his card. Then two special guests will each be given a card and they will each randomly circle 3 numbers on the card. A participant wins a prize if at least one of the numbers he circled is also circled by at least one of the special guests. What is the probability that a participant wins the prize?

11. 在 ΔABC 中，$AB = AC$ 而 $\angle A = 36°$。設 D 為 AC 的垂直平分線與 $\angle B$ 的內角平分線的交點。若 $\angle ADB = x°$，求 x 的值。

In ΔABC, $AB = AC$ and $\angle A = 36°$. Let D be the intersection of the perpendicular bisector of AC and the internal bisector of $\angle B$. If $\angle ADB = x°$, then find the value of x.



12. 若某正整數沒有兩個連續數字相同，且最左和最右的數字也不同，便稱為「好數」，例如：2013 和 2023 是「好數」，2003 和 2012 則不是「好數」。有多少個四位正整數是「好數」？

A positive integer is said to be ‘good’ if no two of its consecutive digits are equal, and that its leftmost digit is different from the rightmost digit. For example, 2013 and 2023 are ‘good’ while 2003 and 2012 are not ‘good’. How many four-digit positive integers are ‘good’?

第 13 至第 16 題，每題 6 分。

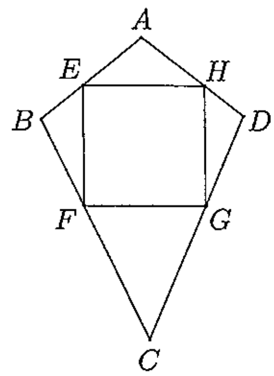
Questions 13 to 16 each carries 6 marks.

13. 現有一個邊長爲 2 的正方形，若從正方形的四個頂點、四邊的中點和正方形中心這九點中隨意選出三個不同的點，這三點能組成一個面積爲正整數的三角形的概率是多少？

There is a square of side length 2. Three different points are randomly chosen from the following nine points: the four vertices of the square, the mid-points of its four sides and its centre. What is the probability that the three chosen points form a triangle whose area is a positive integer?

14. 圖中，$ABCD$ 是鷂形，其中 $AB = AD = 50$ 、 $CB = CD = 75$ 且 $\angle ABC = \angle ADC = 90°$ 。E、F、G、H 分別是 AB、BC、CD、DA 上的點，使得 $EFGH$ 爲正方形，其中 $EF \parallel AC$。求 $EFGH$ 的面積。

In the figure, $ABCD$ is a kite with $AB = AD = 50$, $CB = CD = 75$ and $\angle ABC = \angle ADC = 90°$. E, F, G, H are points on AB, BC, CD, DA respectively such that $EFGH$ is a square with $EF \parallel AC$. Find the area of $EFGH$.

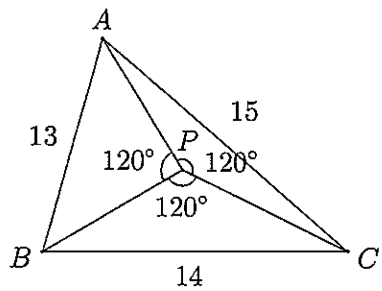


15. 設 $a_0 = 1$，且對任意正整數 n 定義 $a_n = 7^{a_{n-1}}$。求 a_{2013} 的最後兩位數字。

Let $a_0 = 1$ and define $a_n = 7^{a_{n-1}}$ for all positive integers n. Find the last two digits of a_{2013}.

16. 在 ΔABC 中，$AB = 13$ 、 $BC = 14$ 、 $CA = 15$ 。P 爲 ΔABC 內的一點，使得 $\angle APB = \angle BPC = \angle CPA = 120°$ 。求 $(PA + PB + PC)^2$ 的值。

In ΔABC, $AB = 13$, $BC = 14$ and $CA = 15$. P is a point inside ΔABC such that $\angle APB = \angle BPC = \angle CPA = 120°$. Find the value of $(PA + PB + PC)^2$.



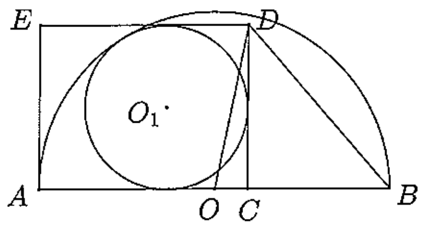
第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 試計算 $\int_0^{\pi} \frac{\cos x + x\sin x - 1}{x^2}\,dx$ 。

Evaluate $\int_0^{\pi} \frac{\cos x + x\sin x - 1}{x^2}\,dx$.

18. 圖中，AB 和 O 分別是半圓的直徑和圓心，且 $AB = 200$ 。以 O_1 爲圓心的小圓內切於半圓，其半徑爲 r，且 $\angle AOO_1 \le 90°$ 。$ACDE$ 是長方形，它的其中三邊均與小圓相切。若 DOB 爲等腰三角形，求 r 所有可能值之和。



In the figure, AB and O are the diameter and centre of the semi-circle respectively, with $AB = 200$. Centred at O_1, the small circle has radius r and is internally tangent to the semi-circle. Also $\angle AOO_1 \le 90°$. $ACDE$ is a rectangle in which three of its sides are tangent to the small circle. If ΔDOB is isosceles, find the sum of all possible values of r.

19. 對非負整數 n，$(1+\sqrt{2}+\sqrt{3})^n$ 可寫成 $a_n + b_n\sqrt{2} + c_n\sqrt{3} + d_n\sqrt{6}$（其中 a_n、b_n、c_n、d_n 爲整數），例如：因爲 $(1+\sqrt{2}+\sqrt{3})^2 = 6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$，故此 $a_2 = 6$ 而 $b_2 = c_2 = d_2 = 2$ 。試在以下方程組中求 x：

$$\begin{cases} a_0x + a_1y + a_2z + a_3w = a_4 \\ b_0x + b_1y + b_2z + b_3w = b_4 \\ c_0x + c_1y + c_2z + c_3w = c_4 \\ d_0x + d_1y + d_2z + d_3w = d_4 \end{cases}$$

For non-negative integer n, $(1+\sqrt{2}+\sqrt{3})^n$ can be written as $a_n + b_n\sqrt{2} + c_n\sqrt{3} + d_n\sqrt{6}$ (where a_n, b_n, c_n and d_n are integers). For example, since $(1+\sqrt{2}+\sqrt{3})^2 = 6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$, we have $a_2 = 6$ and $b_2 = c_2 = d_2 = 2$. Find x in the following system of equations:

$$\begin{cases} a_0x + a_1y + a_2z + a_3w = a_4 \\ b_0x + b_1y + b_2z + b_3w = b_4 \\ c_0x + c_1y + c_2z + c_3w = c_4 \\ d_0x + d_1y + d_2z + d_3w = d_4 \end{cases}$$

20. 已知方程 $x^{20}+x^{13}+3x^{7}+23=0$ 的根分別爲 α_1、α_2、…、α_{20}（不一定是實數）。若方程 $x^{20}+c_{19}x^{19}+c_{18}x^{18}+\cdots+c_1x+c_0=0$（其中 c_0、c_1、…、c_{19} 爲常數）的根分別爲 $\alpha_1^{\,2}$、$\alpha_2^{\,2}$、…、$\alpha_{20}^{\,2}$，求 c_{10} 的值。

It is given that the equation $x^{20}+x^{13}+3x^{7}+23=0$ has roots α_1, α_2, …, α_{20} (not necessarily real). If the equation $x^{20}+c_{19}x^{19}+c_{18}x^{18}+\cdots+c_1x+c_0=0$ (where c_0, c_1, …, c_{19} are constants) has roots $\alpha_1^{\,2}$, $\alpha_2^{\,2}$, …, $\alpha_{20}^{\,2}$, find the value of c_{10}.

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