第十二屆培正數學邀請賽（2013 年）

12th Pui Ching Invitational Mathematics Competition (2013)

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 設 $a*b=(a+b)^2$。求 $(((1*2)*3)*4)*5$ 的最後兩位數字。

   Let $a*b=(a+b)^2$. Find the last two digits of $(((1*2)*3)*4)*5$.

2. 當 345345347349 除以 91 時，餘數是多少？

   What is the remainder when 345345347349 is divided by 91?

3. 當 2013 除以某個兩位正整數時，商和餘數是相同的兩位數。求商。

   When 2013 is divided by a two-digit positive integer, the quotient and remainder are the same two-digit number. Find the quotient.

4. 求 $\frac{2013^3+2012^3}{2013+2012^2}$ 的值。

   Find the value of $\frac{2013^3+2012^3}{2013+2012^2}$.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 若長方形 $ABCD$ 的周界爲 98，而其對角線長度爲 $\sqrt{2013}$，求 $ABCD$ 的面積。

   If the perimeter of rectangle $ABCD$ is 98 and the length of its diagonal is $\sqrt{2013}$, find the area of $ABCD$.

6. 在一個遊戲中，兩位嘉賓分別在 1 和 100 之間（包括首尾兩數）選出 68 個和 45 個整數，然後每位參加者均獲發一張遊戲咭，並可在遊戲咭上填上一個上述範圍內的整數。只要參加者所填寫的數被最少一位嘉賓選中便可中獎。設 p 爲參加者的中獎概率。視乎兩位嘉賓所選的數，p 會取不同的可能值。求 p 的最大和最小可能值之差。

In a game, two guests choose 68 and 45 integers respectively from 1 to 100 (inclusive). Each player is then given a game card on which he may write an integer in the above range. The player can win a prize if the integer written has been chosen by at least one guest. Let p be the probability of winning a prize. The value of p varies according to the integers chosen by the two guests. Find the difference of the largest and smallest possible values of p.

7. 對於正整數 n，設 $f(n)$ 表示 n 的數字之和，例如：因爲 2013 的數字之和是 6，所以 $f(2013)=6$。若 m 是小於 2013 的正整數，且 $[f(m)]^6<m$，問 m 有多少個不同的可能值？

For positive integer n, let $f(n)$ denote the sum of the digits of n. For example, since the sum of the digits of 2013 is 6, we have $f(2013)=6$. If m is a positive integer less than 2013 and $[f(m)]^6<m$, how many different possible values of m are there?

8. 某數學比賽有 20 道題，當中 3、4、5、6、7 分題各有 4 題，合共 100 分。每題答對可得該題所佔的分數，否則該題得 0 分。若加珩在比賽中答對了 18 題並獲得 n 分，求 n 所有可能值之和。

In a mathematical competition there are 20 problems. There are different types of problems carrying 3, 4, 5, 6, 7 marks respectively, with 4 problems for each type, making a total of 100 marks. A correct answer merits all the marks allocated to the problem; otherwise 0 mark is given to the problem. If Ethan answers 18 questions correctly in the competition and gets n marks, find the sum of all possible values of n.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 在所示 n 進制的算式中，每個字母代表一個由 1 至 $(n-1)$ 的不同數字。求 n 的值。

In the addition in base n as shown, each letter represents a different digit from 1 to $(n-1)$. Find the value of n.

$$\begin{array}{r} \text{A B A B} \\ +\quad 4\ 0\ 4\ 4 \\ \hline 1\ 0\ 0\ 0\ \text{A} \end{array}$$

10. 12 張白色咭片被排成一行，現要選其中 4 張塗上黑色，但不能把兩張相鄰的咭片同時塗色。問有多少種不同的塗色方法？

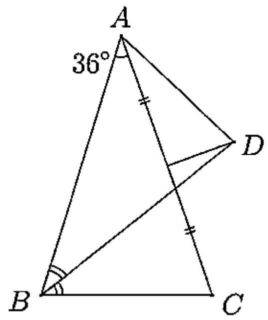
A total of 12 white cards are arranged in a row. Now 4 of them are to be coloured black so that no two adjacent cards are both coloured. How many different ways of colouring are there?

11. 甲、乙兩支足球隊進行互射十二碼比賽，比賽每回合由兩隊輪流派人主射，若兩隊同時射入或同時射失（每次射球只有射入或射失兩種情況），則進入下一回合，直至某回合其中一方射入而另一方射失時，比賽便結束，並由射入的一隊獲勝。已知甲隊每球射入的概率爲 $\frac{2}{3}$ 而乙隊每球射入的概率爲 $\frac{3}{4}$ 。求甲隊在第三回合後便獲勝的概率。

Two football teams, A and B, are having a competition of penalty kicks. In each round, each team would send a member for a shot. If both teams score or both teams miss (each shot either scores or misses), they move on to the next round. The game ends when one team scores while the other misses in a certain round, and the team which scores wins. Given that Team A scores each shot with a probability of $\frac{2}{3}$ while Team B scores each shot with a probability of $\frac{3}{4}$, find the probability that Team A wins immediately after the third round.

12. 在 ΔABC 中，$AB = AC$ 而 $\angle A = 36°$。設 D 爲 AC 的垂直平分線與 $\angle B$ 的內角平分線的交點。若 $\angle ADB = x°$，求 x 的值。

In ΔABC, $AB = AC$ and $\angle A = 36°$. Let D be the intersection of the perpendicular bisector of AC and the internal bisector of $\angle B$. If $\angle ADB = x°$, then find the value of x.



第 13 至第 16 題，每題 6 分。

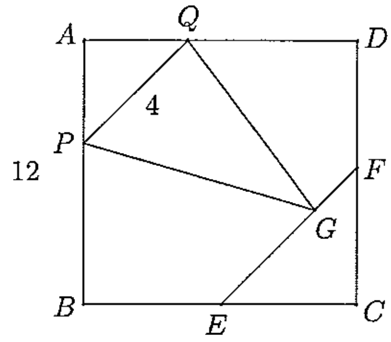
Questions 13 to 16 each carries 6 marks.

13. 五塊面積合共 2700 的長方形紙板被圍成一個無蓋的長方體容器。求該容器的容量的最大可能値。

Five pieces of rectangular cardboard with total area 2700 are made into a container without lid in the shape of a cuboid. Find the largest possible capacity of the container.

14. 圖中，$ABCD$ 是邊長爲 12 的正方形，E 和 F 分別爲 BC 和 CD 的中點；G 是 EF 上的一點，使得 $EG:GF=2:1$；P 和 Q 分別是 AB 和 AD 上的點，使得 $PQ=4$。若四邊形 $APGQ$ 的面積爲 S，求 S^2 的最大可能值。

In the figure, $ABCD$ is a square with side length 12. E and F are the midpoints of sides BC and CD respectively. G is a point on EF such that $EG:GF=2:1$. P and Q are points on sides AB and AD respectively such that $PQ=4$. If the area of quadrilateral $APGQ$ is S, find the greatest possible value of S^2.



15. 求 7^{77777} 的最後六位數字。

Find the last six digits of 7^{77777}.

16. 一個正方體的其中一個頂點被塗上紅色。一隻蟲子從紅色頂點出發，每步牠可從某頂點沿正方體的邊爬行至另一相鄰的頂點（牠可以經過同一頂點超過一次，包括紅色頂點）。若蟲子經過 8 步後返回紅色頂點，則牠所走的路線有多少個不同的可能？

A vertex of a cube is coloured red. A bug starts from the red vertex, and in each step it may crawl from a vertex to an adjacent vertex along the edges of the cube (the same vertex, including the red vertex, can be traversed more than once). If the bug returns to the original vertex after 8 steps, how many different possible routes are there?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 已知方程 $x^{20}+x^{13}+3x^7+23=0$ 的根分別爲 α_1、α_2、…、α_{20}（不一定是實數）。若方程 $x^{20}+c_{19}x^{19}+c_{18}x^{18}+\cdots+c_1x+c_0=0$（其中 c_0、c_1、…、c_{19} 爲常數）的根分別爲 ${\alpha_1}^2$、${\alpha_2}^2$、…、${\alpha_{20}}^2$，求 c_{10} 的值。

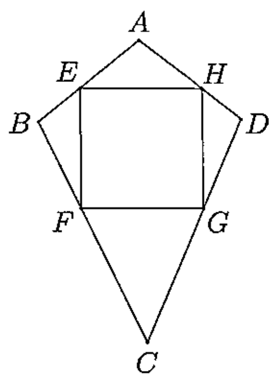
It is given that the equation $x^{20}+x^{13}+3x^7+23=0$ has roots α_1, α_2, …, α_{20} (not necessarily real). If the equation $x^{20}+c_{19}x^{19}+c_{18}x^{18}+\cdots+c_1x+c_0=0$ (where c_0, c_1, …, c_{19} are constants) has roots ${\alpha_1}^2$, ${\alpha_2}^2$, …, ${\alpha_{20}}^2$, find the value of c_{10}.

18. 對於正整數 m、n，設 $f(m,n)$ 表示同時爲 m 的倍數和 n 的因數的正整數數目，例如：因爲同時是 6 的倍數和 24 的因數的正整數有 3 個（分別是 6、12 和 24），所以 $f(6,24)=3$。同時，設 $g(n)=f(1,n)+f(2,n)+\cdots+f(n,n)$。已知 $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$，求 $g(1)+\frac{g(2)}{2^2}+\frac{g(3)}{3^2}+\cdots$ 的值。

For positive integers m, n, let $f(m,n)$ denote the number of positive integers which are both a multiple of m and a factor of n. For example, $f(6,24)=3$ since there are 3 positive integers (namely, 6, 12 and 24) which are both a multiple of 6 and a factor of 24. We also let $g(n)=f(1,n)+f(2,n)+\cdots+f(n,n)$. Given that $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$, find the value of $g(1)+\frac{g(2)}{2^2}+\frac{g(3)}{3^2}+\cdots$.

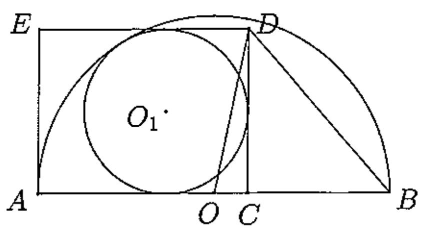
19. 圖中，$ABCD$ 是鷂形，其中 $AB=AD=33$、$CB=CD=80$ 且 $\angle ABC=\angle ADC=90°$。$E$、$F$、$G$、$H$ 分別是 AB、BC、CD、DA 上的點，使得 $EFGH$ 爲長方形（可以是正方形），其中 $EF\,//\,AC$。求 $EFGH$ 的面積的最大值。

In the figure, $ABCD$ is a kite with $AB=AD=33$, $CB=CD=80$ and $\angle ABC=\angle ADC=90°$. E, F, G, H are points on AB, BC, CD, DA respectively such that $EFGH$ is a rectangle (possibly square) with $EF\,//\,AC$. Find the maximum area of $EFGH$.



20. 圖中，AB 和 O 分別是半圓的直徑和圓心，且 $AB=200$。以 O_1 爲圓心的小圓內切於半圓，其半徑爲 r，且 $\angle AOO_1\le 90°$。$ACDE$ 是長方形，它的其中三邊均與小圓相切。若 DOB 爲等腰三角形，求 r 所有可能值之和。

In the figure, AB and O are the diameter and centre of the semi-circle respectively, with $AB=200$. Centred at O_1, the small circle has radius r and is internally tangent to the semi-circle. Also $\angle AOO_1\le 90°$. $ACDE$ is a rectangle in which three of its sides are tangent to the small circle. If ΔDOB is isosceles, find the sum of all possible values of r.



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