第十二屆培正數學邀請賽（2013年）

12th Pui Ching Invitational Mathematics Competition (2013)

決賽（中三組）

Final Event (Secondary 3)

時限：2小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設20題，總分爲100分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案塡在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 若某年的年份中數字互不相同，則稱該年爲「特別年」，例如：1743 年是「特別年」，1744 年則不是「特別年」。若小明出生至今共經歷了三次「特別年」，求他的年齡的最小可能值（年齡以今年的年份減去其出生年份計算）。

   A year is said to be 'special' if all digits of the number of that year are pairwise distinct. For example, the year 1743 is 'special' while the year 1744 is not. If Michael has witnessed three 'special' years since his birth, find the smallest possible value of his age (obtained by subtracting his year of birth from the current year).

2. 當 224991488 除以 149 時，餘數是多少？

   What is the remainder when 224991488 is divided by 149?

3. 黑板上寫著 1 至 2013 這 2013 個正整數。出現次數最少的數字是哪一個？

   On the blackboard the 2013 positive integers from 1 to 2013 are written. Which digit has the lowest number of occurrence?

4. 求 $\frac{2013^3+2012^3}{2013+2012^2}$ 的值。

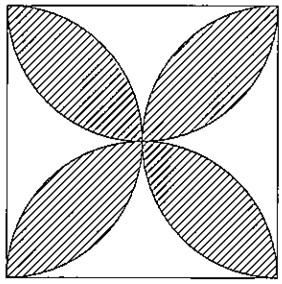
   Find the value of $\frac{2013^3+2012^3}{2013+2012^2}$.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 附圖由一個邊長爲 2 的正方形和四個分別以正方形的四邊爲直徑的半圓弧組成。求陰影部分的面積。

The figure is made up of a square with side length 2 and four semi-circular arcs with the four sides of the square as diameters. Find the area of the shaded region.



6. 已知 $\frac{3}{5}$ 和 $-\frac{5}{8}$ 均爲方程 $120x^3+ax^2+bx-30=0$ 的根。求 ab 的值。

Given that both $\frac{3}{5}$ and $-\frac{5}{8}$ are roots of the equation $120x^3+ax^2+bx-30=0$, find the value of ab.

7. 若長方形 $ABCD$ 的面積爲 $2\sqrt{3}$，而其對角線長度爲 $\sqrt{7}$，求 $ABCD$ 的周界，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 a、b 爲整數。

If the area of rectangle $ABCD$ is $2\sqrt{3}$ and the length of its diagonal is $\sqrt{7}$, find the perimeter of $ABCD$ in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where a and b are integers.

8. 已知 $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$，求 $1+3+6+10+\cdots+1770$（即 1 至 1770 之間所有三角形數之和）的值。

Given that $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$, evaluate $1+3+6+10+\cdots+1770$ (i.e. the sum of all triangular numbers from 1 to 1770).

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字，且 PCIMC、PC 和 MATHDB 所代表的整數都是偶數，它們分別有 16 個、4 個和 40 個正因數。求 MATHDB 所代表的六位數。

In the multiplication shown, each letter represents a different digit from 0 to 9. Furthermore, the integers represented by PCIMC, PC and MATHDB are all even numbers with 16, 4 and 40 positive factors respectively. Find the six-digit number represented by MATHDB.

```
      P C I M C
×           P C
---------------
    M A T H D B
```

10. 設 x 為正數，且 $((x^2+1)^2+16)^2+4^6 \geq 2^{15}+4^7\sqrt{3}$。求 x 的最小可能值，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 a、b 為整數。

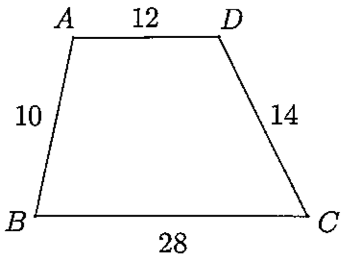
Let x be a positive number such that $((x^2+1)^2+16)^2+4^6 \geq 2^{15}+4^7\sqrt{3}$. Find the smallest possible value of x in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where a and b are integers.

11. 甲、乙兩支足球隊進行互射十二碼比賽，比賽每回合由兩隊輪流派人主射，若兩隊同時射入或同時射失（每次射球只有射入或射失兩種情況），則進入下一回合，直至某回合其中一方射入而另一方射失時，比賽便結束，並由射入的一隊獲勝。已知甲隊每球射入的概率為 $\frac{2}{3}$ 而乙隊每球射入的概率為 $\frac{3}{4}$。求甲隊在第三回合後便獲勝的概率。

Two football teams, A and B, are having a competition of penalty kicks. In each round, each team would send a member for a shot. If both teams score or both teams miss (each shot either scores or misses), they move on to the next round. The game ends when one team scores while the other misses in a certain round, and the team which scores wins. Given that Team A scores each shot with a probability of $\frac{2}{3}$ while Team B scores each shot with a probability of $\frac{3}{4}$, find the probability that Team A wins immediately after the third round.

12. 圖中，$ABCD$ 是梯形，其中 $AD \parallel BC$，$AB=10$，$BC=28$，$CD=14$ 及 $DA=12$。求 $ABCD$ 的面積。

In the figure, $ABCD$ is a trapezium with $AD \parallel BC$, $AB=10$, $BC=28$, $CD=14$ and $DA=12$. Find the area of $ABCD$.



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 現有一個邊長爲 2 的正方形，若從正方形的四個頂點、四邊的中點和正方形中心這九點中隨意選出三個不同的點，這三點能組成一個面積爲正整數的三角形的概率是多少？

There is a square of side length 2. Three different points are randomly chosen from the following nine points: the four vertices of the square, the mid-points of its four sides and its centre. What is the probability that the three chosen points form a triangle whose area is a positive integer?

14. 在一個遊戲中，每名參加者均獲發一張遊戲咭，上面印有 1 至 8 這 8 個數字。參加者需要圈出遊戲咭上的其中 3 個數字。然後兩位特別嘉賓會各自獲發一張遊戲咭，他們每人均會隨意圈出其中 3 個數字。如果某位參加者圈出的每個數字均有最少一位特別嘉賓圈出，該參加者便可獲獎。那麼，一名參加者能在遊戲中獲獎的概率是多少？

In a game, each participant is given a card on which the numbers 1 to 8 are printed. A participant has to circle 3 numbers on his card. Then two special guests will each be given a card and they will each randomly circle 3 numbers on the card. A participant wins a prize if each of the numbers he circled is also circled by at least one of the special guests. What is the probability that a participant wins the prize?

15. 對於正整數 n，設 $f(n)$ 表示 n 的數字之和，例如：因爲 2013 的數字之和是 6，所以 $f(2013)=6$。若 $2f(m^2)+3=f(m)+m$，求 m 所有可能值之和。

For positive integer n, let $f(n)$ denote the sum of the digits of n. For example, since the sum of the digits of 2013 is 6, we have $f(2013)=6$. If $2f(m^2)+3=f(m)+m$, find the sum of all possible values of m.

16. 求 7^{77777} 的最後六位數字。

Find the last six digits of 7^{77777}.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 對於正整數 m、n，設 $f(m,n)$ 表示同時爲 m 的倍數和 n 的因數的正整數數目，例如：因爲同時是 6 的倍數和 24 的因數的正整數有 3 個（分別是 6、12 和 24），所以 $f(6,24)=3$。同時，設 $g(n)=f(1,n)+f(2,n)+\cdots+f(n,n)$。已知 $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$，求 $g(1)+\frac{g(2)}{2^2}+\frac{g(3)}{3^2}+\cdots$ 的值。

    For positive integers m, n, let $f(m,n)$ denote the number of positive integers which are both a multiple of m and a factor of n. For example, $f(6,24)=3$ since there are 3 positive integers (namely, 6, 12 and 24) which are both a multiple of 6 and a factor of 24. We also let $g(n)=f(1,n)+f(2,n)+\cdots+f(n,n)$. Given that $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$, find the value of $g(1)+\frac{g(2)}{2^2}+\frac{g(3)}{3^2}+\cdots$.

18. 某國家有 10 個城市和兩家航空公司。每家航空公司均各自營運航線，每條航線均連接其中兩個城市，並且已知：

    - 任意兩個城市之間，最多只有其中一家航空公司經營一條連接它們的航線；
    - 不存在三個不同的城市，使得當中任意兩者之間均有同一家航空公司的航班連接。

    問兩家航空公司經營的航線合共最多有幾條？

    There are 10 cities and two airlines in a country. Each airline operates flights which connect two of the cities, and it is known that:

    - between any two cities there is at most one flight operated by one of the companies; and
    - there do not exist three different cities such that any two of them are connected by a flight operated by the same airline.

    What is the maximum total number of flights operated by the two airlines?

19. 小陳、小李、小張、小王和小何五人參加了一次象棋聚會，已知他們當中任意兩人對賽最多一次，而且在任意三人之間均可找到曾經對賽和不曾對賽的兩人。事後，他們談論到聚會的情況，其中有些人可能說謊，但已知說真話的人比說謊的人多。

小陳說：「我跟小李和小王都曾經對賽。」

小李說：「我或是跟小王和小何都有對賽，或是跟他們都沒有對賽。」

小張說：「我跟小李曾經對賽。」

小王說：「我沒有跟小何對賽。」

小何說：「另外四人中，所有說謊的人都曾經跟我對賽。」

聚會後，五人按以上順序圍成一圈。他們發現，在聚會的比賽中，由兩個在圓圈上相鄰的人對戰的共有 n 場。若 $m=2^n$ ，求 m 所有可能值之和。

Amy, Betty, Cathy, Danny and Emily joined a chess gathering. It is known that any two of them played at most one game, and that among any three of them one can always find two who have played as well as two who have not played. Afterwards, they talked about the gathering while some of them could be lying. Nevertheless, it is known that more people tell the truth than lie.

Amy said, ‘I have played with both Betty and Danny.’

Betty said, ‘I have played with both Danny and Emily, or neither Danny nor Emily.’

Cathy said, ‘I have played with Betty.’

Danny said, ‘I haven’t played with Emily.’

Emily said, ‘I have played with each of the others who has lied.’

After the gathering, the five people formed a circle in the above order. It was found that of the games played, n of them are between two adjacent people on the circle. If $m=2^n$, find the sum of all possible values of m.

20. 有多少種不同方法可以在圖中的 3×4 方格表中的 12 個小格分別填入 1 至 12，使得任意兩個有公共邊的小格內的兩數之和均爲質數？

In how many different ways can we fill in the numbers 1 to 12 to the 12 cells of the 3×4 grid shown so that the sum of the numbers in any two cells with a common side is a prime number?

全卷完

END OF PAPER