# 第十二屆培正數學邀請賽（2013 年）

# 12th Pui Ching Invitational Mathematics Competition (2013)

## 決賽（中二組）

## Final Event (Secondary 2)

**時限：2 小時**

**Time allowed: 2 hours**

**參賽者須知：**

**Instructions to Contestants:**

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

**第 1 至第 4 題，每題 3 分。**

**Questions 1 to 4 each carries 3 marks.**

1. 在首 2013 個正整數中，可被 3 整除的奇數有多少個？

   Among the first 2013 positive integers, how many odd numbers are divisible by 3?

2. 在 2013 年中，有多少天的「月」和「日」之和是偶數？

   How many days in the year 2013 are there such that the sum of the 'month' and the 'day' is even?

3. 當 224991488 除以 149 時，餘數是多少？

   What is the remainder when 224991488 is divided by 149?

4. 黑板上寫著 1 至 2013 這 2013 個正整數。出現次數最多的數字是哪一個？

   On the blackboard the 2013 positive integers from 1 to 2013 are written. Which digit has the highest number of occurrence?

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**第 5 至第 8 題，每題 4 分。**

**Questions 5 to 8 each carries 4 marks.**

5. 已知 $\frac{3}{5}$ 和 $-\frac{5}{8}$ 均爲方程 $120x^3+ax^2+bx-30=0$ 的根。求 $ab$ 的值。

   Given that both $\frac{3}{5}$ and $-\frac{5}{8}$ are roots of the equation $120x^3+ax^2+bx-30=0$, find the value of $ab$.

6. 設 $a*b=(a+b)^2$。求 $(\cdots(((1*2)*3)*4)*\cdots)*2013$ 的個位數字。

Let $a*b=(a+b)^2$. Find the unit digit of $(\cdots(((1*2)*3)*4)*\cdots)*2013$.

7. 若長方形 $ABCD$ 的面積爲 $2\sqrt{3}$，而其對角線長度爲 $\sqrt{7}$，求 $ABCD$ 的周界，答案以 $a+\sqrt{b}$ 或 $a-\sqrt{b}$ 形式表示，其中 $a$、$b$ 爲整數。

If the area of rectangle $ABCD$ is $2\sqrt{3}$ and the length of its diagonal is $\sqrt{7}$, find the perimeter of $ABCD$ in the form $a+\sqrt{b}$ or $a-\sqrt{b}$ where $a$ and $b$ are integers.

8. 設 $[x]$ 代表不超過 $x$ 的最大整數，例如 $[1.1]=1$、$[6.9]=6$ 和 $[5]=5$。求 $\left[\dfrac{1}{\frac{1}{2001}+\frac{1}{2002}+\cdots+\frac{1}{2013}}\right]$ 的值。

Let $[x]$ denote the greatest integer not exceeding $x$. For example, $[1.1]=1$, $[6.9]=6$ and $[5]=5$. Find the value of $\left[\dfrac{1}{\frac{1}{2001}+\frac{1}{2002}+\cdots+\frac{1}{2013}}\right]$.

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**第 9 至第 12 題，每題 5 分。**

**Questions 9 to 12 each carries 5 marks.**

9. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字，且 PCIMC、PC 和 MATHDB 所代表的整數都是偶數，它們分別有 16 個、4 個和 40 個正因數。求 MATHDB 所代表的六位數。

In the multiplication shown, each letter represents a different digit from 0 to 9. Furthermore, the integers represented by PCIMC, PC and MATHDB are all even numbers with 16, 4 and 40 positive factors respectively. Find the six-digit number represented by MATHDB.

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      P C I M C
×           P C
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    M A T H D B
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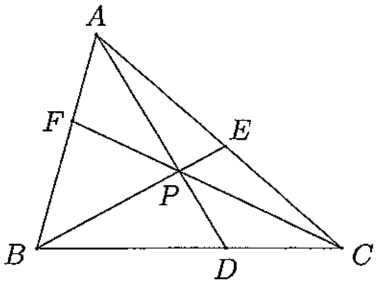
10. 若某正整數的數字中同時有「2」和「3」，則稱爲「好數」，例如：2013 是「好數」，2012 則不是「好數」。有多少個五位數是「好數」？

If a positive integer contains both the digits '2' and '3', it is said to be 'good'. For instance, 2013 is 'good' while 2012 is not. How many five-digit numbers are 'good'?

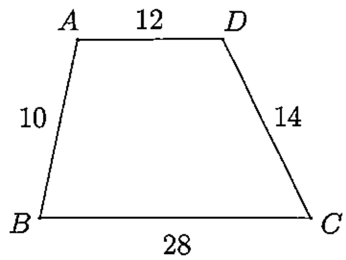
11. 圖中，$D$ 是 $BC$ 上的一點，使得 $BD:DC=2:1$，而 $E$ 則是 $AC$ 的中點。$AD$ 和 $BE$ 交於 $P$，且 $CP$ 延長後交 $AB$ 於 $F$。若 $\Delta ABC$ 的面積為 1，求 $\Delta BDF$ 的面積。

In the figure, $D$ is a point on $BC$ such that $BD:DC=2:1$, while $E$ is the midpoint of $AC$. $AD$ and $BE$ meet at $P$, and the extension of $CP$ meets $AB$ at $F$. If the area of $\Delta ABC$ is 1, find the area of $\Delta BDF$.




12. 圖中，$ABCD$ 是梯形，其中 $AD$ // $BC$，$AB=10$，$BC=28$，$CD=14$ 及 $DA=12$。求 $ABCD$ 的面積。

In the figure, $ABCD$ is a trapezium with $AD$ // $BC$, $AB=10$, $BC=28$, $CD=14$ and $DA=12$. Find the area of $ABCD$.

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**第 13 至第 16 題，每題 6 分。**

**Questions 13 to 16 each carries 6 marks.**

13. 求 $7^{77777}$ 的最後四位數字。

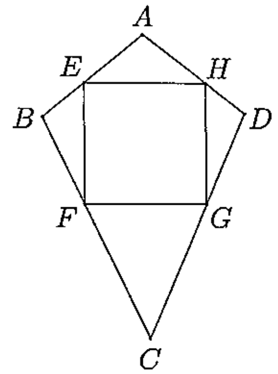
Find the last four digits of $7^{77777}$.

14. 某學校有 2013 名學生，分別編號為 1 至 2013。老師根據學生的編號派發糖果：如果某學生的編號 $m$ 可使得 $m^2-4m+4$ 是 5 的倍數，則該名學生可獲發一顆蘋果味的糖果（例如：因為 $12^2-4(12)+4=100$，是 5 的倍數，故此編號 12 的學生可獲發蘋果味的糖果）；如果某學生的編號 $n$ 可使得 $n^2-3n+6$ 是 8 的倍數，則該名學生可獲發一顆檸檬味的糖果。有多少名學生可以獲發最少一顆糖果？

In a school there are 2013 students, numbered 1 to 2013. The teacher gives out candies according to the student numbers as follows. Each student whose number $m$ is such that $m^2-4m+4$ is a multiple of 5 will be given an apple-flavoured candy (for example, since $12^2-4(12)+4=100$, which is a multiple of 5, the student numbered 12 will be given an apple-flavoured candy), while each student whose number $n$ is such that $n^2-3n+6$ is a multiple of 8 will be given a lemon-flavoured candy. How many students will be given at least one candy?

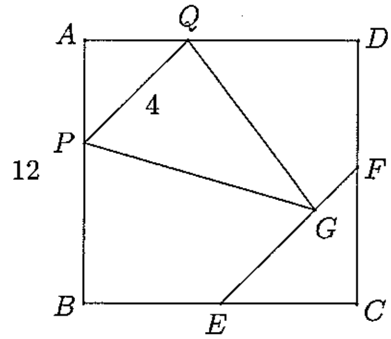
15. 圖中，$ABCD$ 是鷂形，其中 $AB=AD=50$、$CB=CD=75$ 且 $\angle ABC=\angle ADC=90°$。$E$、$F$、$G$、$H$ 分別是 $AB$、$BC$、$CD$、$DA$ 上的點，使得 $EFGH$ 爲正方形，其中 $EF // AC$。求 $EFGH$ 的面積。

In the figure, $ABCD$ is a kite with $AB=AD=50$, $CB=CD=75$ and $\angle ABC=\angle ADC=90°$. $E$, $F$, $G$, $H$ are points on $AB$, $BC$, $CD$, $DA$ respectively such that $EFGH$ is a square with $EF // AC$. Find the area of $EFGH$.




16. 圖中，$ABCD$ 是邊長爲 12 的正方形，$E$ 和 $F$ 分別爲 $BC$ 和 $CD$ 的中點；$G$ 是 $EF$ 上的一點，使得 $EG:GF=2:1$；$P$ 和 $Q$ 分別是 $AB$ 和 $AD$ 上的點，使得 $PQ=4$。若四邊形 $APGQ$ 的面積爲 $S$，求 $S^2$ 的最大可能值。

In the figure, $ABCD$ is a square with side length 12. $E$ and $F$ are the midpoints of sides $BC$ and $CD$ respectively. $G$ is a point on $EF$ such that $EG:GF=2:1$. $P$ and $Q$ are points on sides $AB$ and $AD$ respectively such that $PQ=4$. If the area of quadrilateral $APGQ$ is $S$, find the greatest possible value of $S^2$.

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**第 17 至第 20 題，每題 7 分。**

**Questions 17 to 20 each carries 7 marks.**

17. 對於正整數 $m$、$n$，設 $f(m,n)$ 表示同時爲 $m$ 的倍數和 $n$ 的因數的正整數數目，例如：因爲同時是 6 的倍數和 24 的因數的正整數有 3 個（分別是 6、12 和 24），所以 $f(6,24)=3$。同時，設 $g(n)=f(1,n)+f(2,n)+\cdots+f(n,n)$。已知 $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$，求 $g(1)+\frac{g(2)}{2^2}+\frac{g(3)}{3^2}+\cdots$ 的值。

For positive integers $m$, $n$, let $f(m,n)$ denote the number of positive integers which are both a multiple of $m$ and a factor of $n$. For example, $f(6,24)=3$ since there are 3 positive integers (namely, 6, 12 and 24) which are both a multiple of 6 and a factor of 24. We also let $g(n)=f(1,n)+f(2,n)+\cdots+f(n,n)$. Given that $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$, find the value of $g(1)+\frac{g(2)}{2^2}+\frac{g(3)}{3^2}+\cdots$.

18. 某國家有 10 個城市和兩家航空公司。每家航空公司均各自營運航線，每條航線均連接其中兩個城市，並且已知：

- 任意兩個城市之間，最多只有其中一家航空公司經營一條連接它們的航線；
- 不存在三個不同的城市，使得當中任意兩者之間均有同一家航空公司的航班連接。

問兩家航空公司經營的航線合共最多有幾條？

There are 10 cities and two airlines in a country. Each airline operates flights which connect two of the cities, and it is known that:

- between any two cities there is at most one flight operated by one of the companies; and
- there do not exist three different cities such that any two of them are connected by a flight operated by the same airline.

What is the maximum total number of flights operated by the two airlines?

19. 對非負整數 $n$，$(1+\sqrt{2}+\sqrt{3})^n$ 可寫成 $a_n+b_n\sqrt{2}+c_n\sqrt{3}+d_n\sqrt{6}$（其中 $a_n$、$b_n$、$c_n$、$d_n$ 爲整數），例如：因爲 $(1+\sqrt{2}+\sqrt{3})^2=6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$，故此 $a_2=6$ 而 $b_2=c_2=d_2=2$。試在以下方程組中求 $x$：

$$\begin{cases} a_0x+a_1y+a_2z+a_3w=a_4 \\ b_0x+b_1y+b_2z+b_3w=b_4 \\ c_0x+c_1y+c_2z+c_3w=c_4 \\ d_0x+d_1y+d_2z+d_3w=d_4 \end{cases}$$

For non-negative integer $n$, $(1+\sqrt{2}+\sqrt{3})^n$ can be written as $a_n+b_n\sqrt{2}+c_n\sqrt{3}+d_n\sqrt{6}$ (where $a_n$, $b_n$, $c_n$ and $d_n$ are integers). For example, since $(1+\sqrt{2}+\sqrt{3})^2=6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$, we have $a_2=6$ and $b_2=c_2=d_2=2$. Find $x$ in the following system of equations:

$$\begin{cases} a_0x+a_1y+a_2z+a_3w=a_4 \\ b_0x+b_1y+b_2z+b_3w=b_4 \\ c_0x+c_1y+c_2z+c_3w=c_4 \\ d_0x+d_1y+d_2z+d_3w=d_4 \end{cases}$$

20. 小陳、小李、小張、小王和小何五人參加了一次象棋聚會，已知他們當中任意兩人對賽最多一次，而且在任意三人之間均可找到曾經對賽和不曾對賽的兩人。事後，他們談論到聚會的情況，其中有些人可能說謊，但已知說真話的人比說謊的人多。

小陳說：「我跟小李和小王都曾經對賽。」

小李說：「我或是跟小王和小何都有對賽，或是跟他們都沒有對賽。」

小張說：「我跟小李曾經對賽。」

小王說：「我沒有跟小何對賽。」

小何說：「另外四人中，所有說謊的人都曾經跟我對賽。」

聚會後，五人按以上順序圍成一圈。他們發現，在聚會的比賽中，由兩個在圓圈上相鄰的人對戰的共有 $n$ 場。若 $m = 2^n$，求 $m$ 所有可能值之和。

Amy, Betty, Cathy, Danny and Emily joined a chess gathering. It is known that any two of them played at most one game, and that among any three of them one can always find two who have played as well as two who have not played. Afterwards, they talked about the gathering while some of them could be lying. Nevertheless, it is known that more people tell the truth than lie.

Amy said, 'I have played with both Betty and Danny.'

Betty said, 'I have played with both Danny and Emily, or neither Danny nor Emily.'

Cathy said, 'I have played with Betty.'

Danny said, 'I haven't played with Emily.'

Emily said, 'I have played with each of the others who has lied.'

After the gathering, the five people formed a circle in the above order. It was found that of the games played, $n$ of them are between two adjacent people on the circle. If $m = 2^n$, find the sum of all possible values of $m$.

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全卷完

**END OF PAPER**