第十二屆培正數學邀請賽（2013 年）

12th Pui Ching Invitational Mathematics Competition (2013)

決賽（中一組）

Final Event (Secondary 1)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 求最小的正整數 n，使得 $1+2+\cdots+n$ 大於 2013。

   Find the smallest positive integer n such that $1+2+\cdots+n$ is larger than 2013.

2. 若某年的年份中數字互不相同，則稱該年為「特別年」，例如：1743 年是「特別年」，1744 年則不是「特別年」。若小明出生至今共經歷了三次「特別年」，求他的年齡的最小可能值（年齡以今年的年份減去其出生年份計算）。

   A year is said to be 'special' if all digits of the number of that year are pairwise distinct. For example, the year 1743 is 'special' while the year 1744 is not. If Michael has witnessed three 'special' years since his birth, find the smallest possible value of his age (obtained by subtracting his year of birth from the current year).

3. 設 $a*b=(a+b)^2$。求 $(((1*2)*3)*4)*5$ 的個位數字。

   Let $a*b=(a+b)^2$. Find the unit digit of $(((1*2)*3)*4)*5$.

4. 當 911911911911 除以 91 時，餘數是多少？

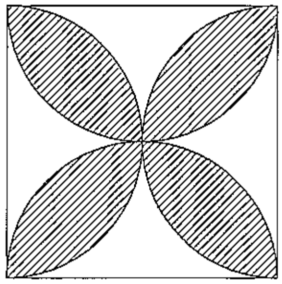
   What is the remainder when 911911911911 is divided by 91?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 附圖由一個邊長爲 2 的正方形和四個分別以正方形的四邊爲直徑的半圓弧組成。求陰影部分的面積。

The figure is made up of a square with side length 2 and four semi-circular arcs with the four sides of the square as diameters. Find the area of the shaded region.



6. 某數學比賽有 10 道題，前 6 題每題 8 分，後 4 題每題 13 分。每題答對可得該題所佔的分數，否則該題得 0 分。那麼，參賽者在比賽中的得分有多少個不同的可能值？

In a mathematical competition there are 10 problems. The first 6 problems carry 8 marks each while the last 4 problems carry 13 marks each. A correct answer merits all the marks allocated to the problem; otherwise 0 mark is given to the problem. How many different possible scores can a contestant get?

7. 已知 $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$，求 $1+3+6+10+\cdots+1770$（即 1 至 1770 之間所有三角形數之和）的值。

Given that $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$, evaluate $1+3+6+10+\cdots+1770$ (i.e. the sum of all triangular numbers from 1 to 1770).

8. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1]=1$、$[6.9]=6$ 和 $[5]=5$。求 $\left[\frac{1}{\frac{1}{2001}+\frac{1}{2002}+\cdots+\frac{1}{2013}}\right]$ 的值。

Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1]=1$, $[6.9]=6$ and $[5]=5$. Find the value of $\left[\frac{1}{\frac{1}{2001}+\frac{1}{2002}+\cdots+\frac{1}{2013}}\right]$.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字，且 PCIMC、PC 和 MATHDB 所代表的整數都是偶數，它們分別有 16 個、4 個和 40 個正因數。求 MATHDB 所代表的六位數。

In the multiplication shown, each letter represents a different digit from 0 to 9. Furthermore, the integers represented by PCIMC, PC and MATHDB are all even numbers with 16, 4 and 40 positive factors respectively. Find the six-digit number represented by MATHDB.

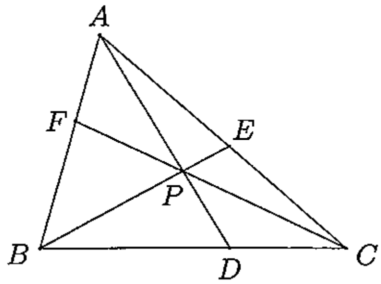
$$\begin{array}{r} \mathrm{P\ C\ I\ M\ C} \\ \times \qquad\qquad \mathrm{P\ C} \\ \hline \mathrm{M\ A\ T\ H\ D\ B} \end{array}$$

10. 若某正整數的數字中同時有「2」和「3」，則稱爲「好數」，例如：2013 是「好數」，2012 則不是「好數」。有多少個五位數是「好數」？

If a positive integer contains both the digits ‘2’ and ‘3’, it is said to be ‘good’. For instance, 2013 is ‘good’ while 2012 is not. How many five-digit numbers are ‘good’?

11. 圖中，D 是 BC 上的一點，使得 $BD:DC=2:1$，而 E 則是 AC 的中點。AD 和 BE 交於 P，且 CP 延長後交 AB 於 F。若 ΔABC 的面積爲 1，求 ΔBDF 的面積。

In the figure, D is a point on BC such that $BD:DC=2:1$, while E is the midpoint of AC. AD and BE meet at P, and the extension of CP meets AB at F. If the area of ΔABC is 1, find the area of ΔBDF .



12. 某凸 n 邊形的所有內角（以「度」位單位時）均爲整數，且互不相等。求 n 的所有可能值之和。

All interior angles of a convex n-gon are integers (in degrees) and are pairwise distinct. Find the sum of all possible values of n.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 某學校有 2013 名學生，分別編號爲 1 至 2013。老師根據學生的編號派發糖果：如果某學生的編號 m 可使得 m^2-4m+4 是 5 的倍數，則該名學生可獲發一顆蘋果味的糖果（例如：因爲 $12^2-4(12)+4=100$，是 5 的倍數，故此編號 12 的學生可獲發蘋果味的糖果）；如果某學生的編號 n 可使得 n^2-3n+6 是 8 的倍數，則該名學生可獲發一顆檸檬味的糖果。有多少名學生可以獲發最少一顆糖果？

    In a school there are 2013 students, numbered 1 to 2013. The teacher gives out candies according to the student numbers as follows. Each student whose number m is such that m^2-4m+4 is a multiple of 5 will be given an apple-flavoured candy (for example, since $12^2-4(12)+4=100$, which is a multiple of 5, the student numbered 12 will be given an apple-flavoured candy), while each student whose number n is such that n^2-3n+6 is a multiple of 8 will be given a lemon-flavoured candy. How many students will be given at least one candy?

14. 求 7^{77777} 的最後四位數字。

    Find the last four digits of 7^{77777}.

15. 對於正整數 n，設 $f(n)$ 表示 n 的數字之和，例如：因爲 2013 的數字之和是 6，所以 $f(2013)=6$。若 $2f(m^2)+3=f(m)+m$，求 m 所有可能値之和。

    For positive integer n, let $f(n)$ denote the sum of the digits of n. For example, since the sum of the digits of 2013 is 6, we have $f(2013)=6$. If $2f(m^2)+3=f(m)+m$, find the sum of all possible values of m.

16. 一個正方體的其中一個頂點被塗上紅色。一隻蟲子從紅色頂點出發，每步牠可從某頂點沿正方體的邊爬行至另一相鄰的頂點（牠可以經過同一頂點超過一次，包括紅色頂點）。若蟲子經過 8 步後返回紅色頂點，則牠所走的路線有多少個不同的可能？

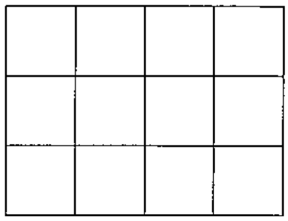
    A vertex of a cube is coloured red. A bug starts from the red vertex, and in each step it may crawl from a vertex to an adjacent vertex along the edges of the cube (the same vertex, including the red vertex, can be traversed more than once). If the bug returns to the original vertex after 8 steps, how many different possible routes are there?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 有多少種不同方法可以在圖中的 3×4 方格表中的 12 個小格分別填入 1 至 12，使得任意兩個有公共邊的小格內的兩數之和均爲質數？

    In how many different ways can we fill in the numbers 1 to 12 to the 12 cells of the 3×4 grid shown so that the sum of the numbers in any two cells with a common side is a prime number?



18. 小陳、小李、小張、小王和小何五人參加了一次象棋聚會，已知他們當中任意兩人對賽最多一次，而且在任意三人之間均可找到曾經對賽和不曾對賽的兩人。事後，他們談論到聚會的情況，其中有些人可能說謊，但已知說真話的人比說謊的人多。

    小陳說：「我跟小李和小王都曾經對賽。」

    小李說：「我或是跟小王和小何都有對賽，或是跟他們都沒有對賽。」

    小張說：「我跟小李曾經對賽。」

    小王說：「我沒有跟小何對賽。」

    小何說：「另外四人中，所有說謊的人都曾經跟我對賽。」

    聚會後，五人按以上順序圍成一圈。他們發現，在聚會的比賽中，由兩個在圓圈上相鄰的人對戰的共有 n 場。若 $m=2^n$，求 m 所有可能值之和。

    Amy, Betty, Cathy, Danny and Emily joined a chess gathering. It is known that any two of them played at most one game, and that among any three of them one can always find two who have played as well as two who have not played. Afterwards, they talked about the gathering while some of them could be lying. Nevertheless, it is known that more people tell the truth than lie.

    Amy said, 'I have played with both Betty and Danny.'

    Betty said, 'I have played with both Danny and Emily, or neither Danny nor Emily.'

    Cathy said, 'I have played with Betty.'

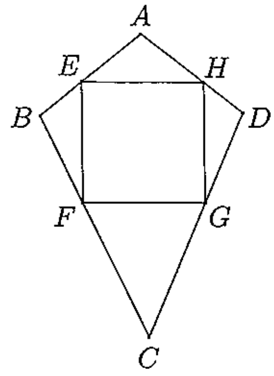
    Danny said, 'I haven't played with Emily.'

    Emily said, 'I have played with each of the others who has lied.'

    After the gathering, the five people formed a circle in the above order. It was found that of the games played, n of them are between two adjacent people on the circle. If $m=2^n$, find the sum of all possible values of m.

19. 圖中，$ABCD$ 是鷂形，其中 $AB=AD=33$、$CB=CD=80$ 且 $\angle ABC=\angle ADC=90°$。$E$、$F$、$G$、$H$ 分別是 AB、BC、CD、DA 上的點，使得 $EFGH$ 爲長方形（可以是正方形），其中 $EF\,//\,AC$。求 $EFGH$ 的面積的最大值。

In the figure, $ABCD$ is a kite with $AB=AD=33$, $CB=CD=80$ and $\angle ABC=\angle ADC=90°$. E, F, G, H are points on AB, BC, CD, DA respectively such that $EFGH$ is a rectangle (possibly square) with $EF\,//\,AC$. Find the maximum area of $EFGH$.



20. 對非負整數 n，$(1+\sqrt{2}+\sqrt{3})^n$ 可寫成 $a_n+b_n\sqrt{2}+c_n\sqrt{3}+d_n\sqrt{6}$（其中 a_n、b_n、c_n、d_n 爲整數），例如：因爲 $(1+\sqrt{2}+\sqrt{3})^2=6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$，故此 $a_2=6$ 而 $b_2=c_2=d_2=2$。試在以下方程組中求 x：

$$\begin{cases} a_0x+a_1y+a_2z+a_3w=a_4 \\ b_0x+b_1y+b_2z+b_3w=b_4 \\ c_0x+c_1y+c_2z+c_3w=c_4 \\ d_0x+d_1y+d_2z+d_3w=d_4 \end{cases}$$

For non-negative integer n, $(1+\sqrt{2}+\sqrt{3})^n$ can be written as $a_n+b_n\sqrt{2}+c_n\sqrt{3}+d_n\sqrt{6}$ (where a_n, b_n, c_n and d_n are integers). For example, since $(1+\sqrt{2}+\sqrt{3})^2=6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$, we have $a_2=6$ and $b_2=c_2=d_2=2$. Find x in the following system of equations:

$$\begin{cases} a_0x+a_1y+a_2z+a_3w=a_4 \\ b_0x+b_1y+b_2z+b_3w=b_4 \\ c_0x+c_1y+c_2z+c_3w=c_4 \\ d_0x+d_1y+d_2z+d_3w=d_4 \end{cases}$$

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