第十一屆培正數學邀請賽（**2012** 年）

11th Pui Ching Invitational Mathematics Competition (2012)

決賽（高中組）

Final Event (Senior Secondary)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

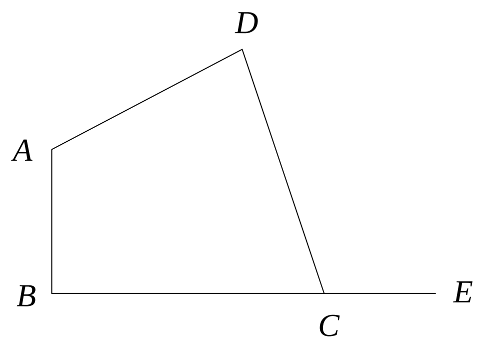
Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 圖中，$ABCD$ 是圓內接四邊形，BCE 是直線。若 $AD = BC$ 、$\angle DCE = 123°$ 而 $\angle ADC = x°$，求 x。

   In the figure, $ABCD$ is a cyclic quadrilateral and BCE is a straight line. If $AD = BC$, $\angle DCE = 123°$ and $\angle ADC = x°$, find x.



2. 設 A、B、C 爲某概率空間中的三件事件，其中 A 和 C 是互斥事件，而 B 和 C 則是獨立事件。已知 $P(A) = 0.2012$ 、$P(B) = 0.05$ 和 $P(C) = 0.12$。若 $D = A \cup B$，求 $P(D' \mid C)$，其中 D' 表示 D 的互補事件。

   Let A, B, C be events in a probability space, where the events A and C are mutually exclusive, while the events B and C are independent. Given $P(A) = 0.2012$, $P(B) = 0.05$ and $P(C) = 0.12$. If $D = A \cup B$, find $P(D' \mid C)$, where D' denotes the complementary event of D.

3. 設 a、b 爲非零常數。若方程 $ax^2 + (a+b)x + b = 0$ 有重根 m，求 m。

   Let a, b be non-zero constants. If the equation $ax^2 + (a+b)x + b = 0$ has a repeated root m, find m.

4. 若 $\sqrt{\dfrac{x-14}{x+13}} + \sqrt{\dfrac{x+13}{x-14}} = \dfrac{41}{20}$，其中 $x > 0$，求 x。

   If $\sqrt{\dfrac{x-14}{x+13}} + \sqrt{\dfrac{x+13}{x-14}} = \dfrac{41}{20}$ where $x > 0$, find x.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 求方程 $\dfrac{2012^3+2102^3}{x^3+2012^3}=\dfrac{4114}{x+2012}$ 的最小實根。

   Find the smallest real root to the equation $\dfrac{2012^3+2102^3}{x^3+2012^3}=\dfrac{4114}{x+2012}$.

6. 設 a、b、c、d、e 爲非零常數。若方程 $ax^4+bx^3+cx^2+dx+e=0$ 有剛好 2 個實數解，則方程 $ex^5+dx^4+cx^3+bx^2+ax=0$ 有多少個實數解？

   Let a, b, c, d, e be non-zero constants. If the equation $ax^4+bx^3+cx^2+dx+e=0$ has exactly 2 real solutions, how many real solutions are there for the equation $ex^5+dx^4+cx^3+bx^2+ax=0$?

7. 在所示的算式中，每個字母代表一個由 0 至 9 的數字（不同字母代表的數字有可能相同）。求乘積 ABBA 所代表的四位數。

   In the multiplication shown, each letter represents a digit from 0 to 9 (different letters may represent the same digit). Find the four-digit number represented by the product ABBA.

$$\begin{array}{r} \mathrm{B\ C\ B} \\ \times\quad \mathrm{D\ B} \\ \hline \mathrm{A\ B\ B\ A} \end{array}$$

8. 若正整數 n 有三位或以上，首尾兩位數字相同，且其餘數字均與首尾兩位數字不同，則 n 稱爲「好數」。例如：1231 和 2332 是「好數」，1234 和 15611 則不是「好數」。有多少個「好數」不大於 2012？

   If a positive integer n has three or more digits, has the first and last digits equal and all other digits different from the first and last digits, then we say that n is a ‘good’ integer. For example, 1231 and 2332 are ‘good’ integers, while 1234 and 15611 are not ‘good’. How many ‘good’ integers not exceeding 2012 are there?

第 9 至第 12 題，每題 5 分。

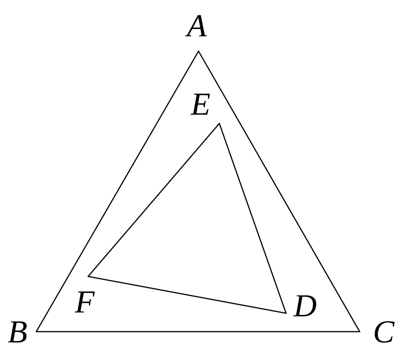
Questions 9 to 12 each carries 5 marks.

9. 已知 $p=a^2-b^2$，其中 p 是小於 100 的質數，而 a、b 則爲整數。在 a 和 b 兩個整數中，其中一個是 2 的倍數，另一個是 3 的倍數。那麼 p 有多少個不同的可能值？

   Given $p=a^2-b^2$, where p is a prime number less than 100, while a and b are integers. Of the two integers a and b, one is divisible by 2 and the other is divisible by 3. How many different possible values of p are there?

11. 圖中，ABC 和 DEF 皆是等邊三角形，邊長分別是 13 和 7，其中 ΔDEF 完全位於 ΔABC 內，且兩個三角形的中心相同。求 AD^2 的最大可能值。

    In the figure, ABC and DEF are equilateral triangles with side length 13 and 7 respectively, such that ΔDEF lies wholly within ΔABC and the two triangles have the same centre. Find the maximum possible value of AD^2.

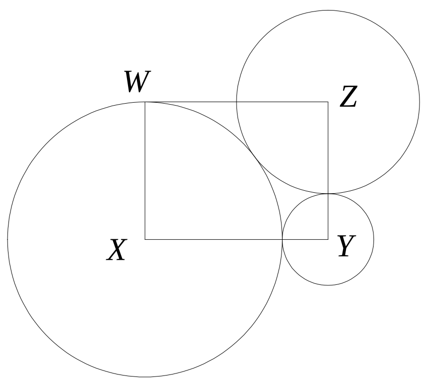


11. 方程 $xyz=1400$ 有多少組正整數解？

    How many sets of positive integer solutions are there to the equation $xyz=1400$?

12. 如圖所示，三個分別以 X、Y 和 Z 爲圓心的圓互相外切。W 是以 X 爲圓心的圓上的一點，使得 $XYZW$ 爲長方形。若長方形的對角線長度爲 20，求長方形的周界。

    In the figure, three circles centred at X, Y and Z respectively are externally tangent to each other. W is a point on the circle centred at X such that $XYZW$ is a rectangle. If the length of a diagonal of the rectangle is 20, find the perimeter of the rectangle.



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 求 54321^{43210} 的最後五位數字。

    Find the last five digits of 54321^{43210}.

14. 在一個 7×7 方格表的每格中均有一個正整數。已知整個表格中共有 k 個不同的正整數，且當兩格有公共頂點時，該兩格內的兩數之和均爲質數。求 k 的最大可能值。

    In each cell of a 7×7 table there is a positive integer. It is known that there are altogether k different positive integers in the entire table, and that whenever two cells share a common vertex, the sum of their two numbers must be prime. Find the greatest possible value of k.

15. 已知函數 f 滿足以下條件：

    - 對任意實數 x， $f(x)$ 、 $f'(x)$ 和 $f''(x)$ 均存在且爲實數；
    - 對任意實數 x 皆有 $f'(x)\le f(x)$ ；
    - 對任意 $x\ge 0$ 皆有 $f(x)=0$ ；
    - 對任意 $x<0$ 皆有 $f(x)=Ax^n$ ，其中 A 是固定的正常數，n 是不超過 100 的固定正整數。

    求 n 所有可能值之和。

    Given that the function f satisfies the following conditions:

    - $f(x)$, $f'(x)$ and $f''(x)$ exist and are real for any real number x;
    - $f'(x)\le f(x)$ for any real number x;
    - $f(x)=0$ for all $x\ge 0$;
    - $f(x)=Ax^n$ for all $x<0$, where A is a fixed positive constant and n is a fixed positive integer not exceeding 100.

    Find the sum of all possible values of n.

16. 設 $a_1=1$ ，且對正整數 n 定義 $a_{n+1}=3a_1a_2\cdots a_n+2$ 。那麼正整數 a_{10} 有多少個位？

    Let $a_1=1$ and define $a_{n+1}=3a_1a_2\cdots a_n+2$ for positive integer n. How many digits are there in the positive integer a_{10}?

第 17 至第 20 題，每題 7 分。

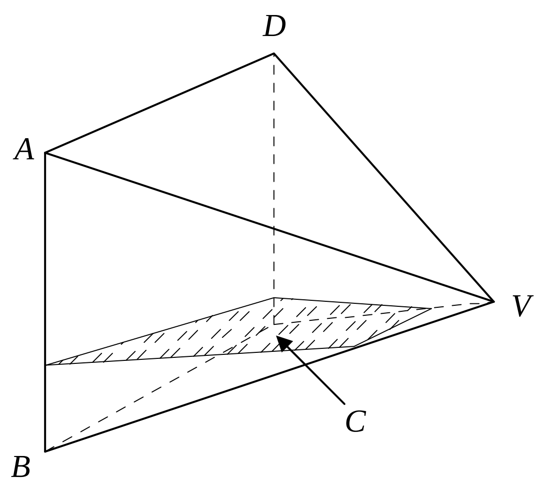
Questions 17 to 20 each carries 7 marks.

17. 設 a、x 爲實數，使得 $17x^4+34ax^3+(25a^2+10)x^2+(8a^3+10a)x+(a^4+25)=0$。求 a 所有可能值之積。

Let a, x be real numbers such that $17x^4+34ax^3+(25a^2+10)x^2+(8a^3+10a)x+(a^4+25)=0$. Find the product of all possible values of a.

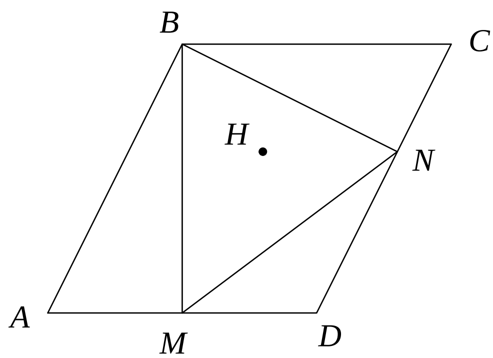
18. 一些水被倒進錐體 $VABCD$，其底 $ABCD$ 是邊長爲 10 的正方形，而 $VA=VB=VC=VD=20$。如圖所示，現放置錐體使得 BC 在地面上，而 AB 和 DC 均爲鉛垂。若水位離地面的高度爲 3 而水的體積爲 V，求 V^2。

Water is poured into the pyramid $VABCD$ in which the square base $ABCD$ has side length 10 and $VA=VB=VC=VD=20$. The pyramid is now placed so that BC is on the ground while AB and DC are vertical, as shown in the figure. If the height of the water level from the ground is 3 and the volume of water is V, find V^2.



19. 圖中，$ABCD$ 是平行四邊形，$\angle ABC$ 是鈍角。M 和 N 分別是 B 到 AD 和 CD 的垂足，而 H 則是 ΔBMN 的垂心。若 $MN = 3$ 而 $BD = 4$，求 BH。

In the figure, $ABCD$ is a parallelogram and $\angle ABC$ is obtuse. M and N are the feet of perpendiculars from B to AD and CD respectively, while H is the orthocentre of ΔBMN. If $MN = 3$ and $BD = 4$, find BH.



20. 設 $f(x)=1-|1-2x|$，且 $0\le k\le 1$。若無窮數列 k、$f(k)$、$f(f(k))$、$f(f(f(k)))$、⋯ 中剛好有 6 個不同的數，求 k 的最大可能值。

Let $f(x)=1-|1-2x|$ and $0\le k\le 1$. If there are exactly 6 different numbers in the infinite sequence k, $f(k)$, $f(f(k))$, $f(f(f(k)))$, …, find the greatest possible value of k.

全卷完

END OF PAPER