第十一屆培正數學邀請賽（**2012** 年）

11th Pui Ching Invitational Mathematics Competition (2012)

決賽（中四組）

Final Event (Secondary 4)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 設 A、B、C 爲某概率空間中的三件事件，其中 A 和 C 是互斥事件，而 B 和 C 則是獨立事件。已知 $P(A)=0.2012$ 、 $P(B)=0.05$ 和 $P(C)=0.12$。若 $D=A\cup B$ ，求 $P(D'|C)$ ，其中 D' 表示 D 的互補事件。

   Let A, B, C be events in a probability space, where the events A and C are mutually exclusive, while the events B and C are independent. Given $P(A)=0.2012$, $P(B)=0.05$ and $P(C)=0.12$. If $D=A\cup B$, find $P(D'|C)$, where D' denotes the complementary event of D.

2. 設 a、b 爲非零常數。若方程 $ax^2+(a+b)x+b=0$ 有重根 m，求 m。

   Let a, b be non-zero constants. If the equation $ax^2+(a+b)x+b=0$ has a repeated root m, find m.

3. 設 a、b 爲整數，使得 $(a-bi)(a+bi)=5$。求 $a+b$ 的最小可能值。

   Let a, b be integers such that $(a-bi)(a+bi)=5$. Find the smallest possible value of $a+b$.

4. 在一個 3×3 方格表中，每格均有一個整數，使得每個橫行、每個直行和每條對角線上三個數之和均等於 S。附圖顯示了其中三格中的整數。求 S 的值。

   In a 3×3 table, there is an integer inside each cell so that the sum of the three numbers in each row, in each column and on each diagonal is equal to S. The figure shows the integers in three of the cells. Find the value of S.

12		
		5
		12

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 設 k 為常數。若對任意實數 a，方程 $x^2-2ax+16a-k-9=0$ 皆有實根，求 k 的最小可能值。

   Let k be a constant. If the equation $x^2-2ax+16a-k-9=0$ has a real root for any real number a, find the smallest possible value of k.

6. 在所示的算式中，每個字母代表一個由 0 至 9 的數字（不同字母代表的數字有可能相同）。求乘積 ABBA 所代表的四位數。

   In the multiplication shown, each letter represents a digit from 0 to 9 (different letters may represent the same digit). Find the four-digit number represented by the product ABBA.

$$\begin{array}{r} \text{B C B} \\ \times \quad \text{D B} \\ \hline \text{A B B A} \end{array}$$

7. 若正整數 n 有三位或以上，首尾兩位數字相同，且其餘數字均與首尾兩位數字不同，則 n 稱為「好數」。例如：1231 和 2332 是「好數」，1234 和 15611 則不是「好數」。有多少個「好數」不大於 2012？

   If a positive integer n has three or more digits, has the first and last digits equal and all other digits different from the first and last digits, then we say that n is a ‘good’ integer. For example, 1231 and 2332 are ‘good’ integers, while 1234 and 15611 are not ‘good’. How many ‘good’ integers not exceeding 2012 are there?

8. 某次數學競賽有一道填充題：「對任意的正整數 a、b，若 $ab+1$ 是 16 的倍數，則 $a+b$ 必定是 ____ 的倍數。」這題有很多不同的正確答案。求最大的一個正確答案。

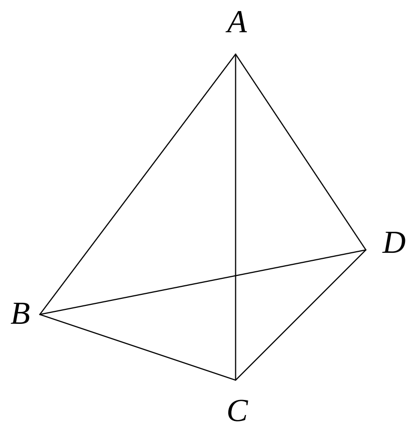
   In a mathematical competition there was a fill-in-the-blank question: ‘For any positive integers a and b, if $ab+1$ is a multiple of 16, then $a+b$ must be a multiple of ____.’ There are many correct answers to this question. Find the largest correct answer.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 圖中的四面體框架 $ABCD$ 由六枝木棒組成，其中 $AC = 9$、$BC = 7$、$BD = 10$。已知可以把一個適當大小的球體放進框架中，使得球體與每枝木棒都有接觸。求 AD。

In the figure, the tetrahedral frame $ABCD$ is made up of six wooden sticks in which $AC = 9$, $BC = 7$ and $BD = 10$. It is known that one can put a sphere of suitable size into the frame so that it touches every wooden stick. Find AD.

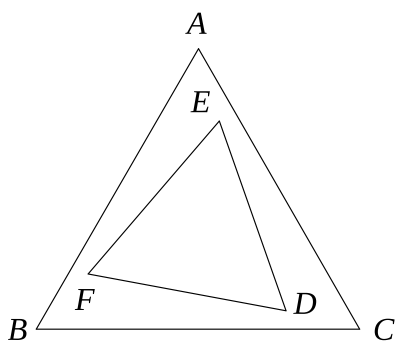


10. 已知 $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$，求 $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$。

Given that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$, find $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$.

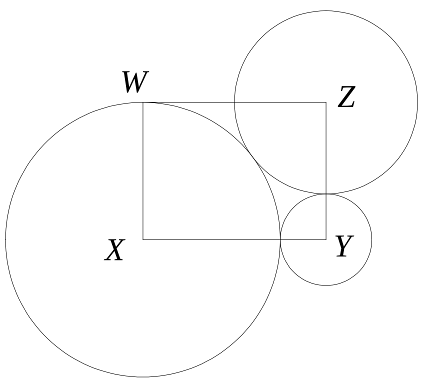
11. 圖中，ABC 和 DEF 皆是等邊三角形，邊長分別是 13 和 7，其中 ΔDEF 完全位於 ΔABC 內，且兩個三角形的中心相同。求 AD^2 的最大可能值。

In the figure, ABC and DEF are equilateral triangles with side length 13 and 7 respectively, such that ΔDEF lies wholly within ΔABC and the two triangles have the same centre. Find the maximum possible value of AD^2.



12. 如圖所示，三個分別以 X、Y 和 Z 爲圓心的圓互相外切。W 是以 X 爲圓心的圓上的一點，使得 $XYZW$ 爲長方形。若長方形的對角線長度爲 20，求長方形的周界。

In the figure, three circles centred at X, Y and Z respectively are externally tangent to each other. W is a point on the circle centred at X such that $XYZW$ is a rectangle. If the length of a diagonal of the rectangle is 20, find the perimeter of the rectangle.



第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 設 $a_1=1$，且對正整數 n 定義 $a_{n+1}=3a_1a_2\cdots a_n+2$。那麼正整數 a_{10} 有多少個位？

 Let $a_1=1$ and define $a_{n+1}=3a_1a_2\cdots a_n+2$ for positive integer n. How many digits are there in the positive integer a_{10}?

14. 若實數 x 滿足方程 $6x=8^x$，求 x 所有可能值之積。

 If the real number x satisfies the equation $6x=8^x$, find the product of all possible values of x.

15. 求 54321^{43210} 的最後五位數字。

 Find the last five digits of 54321^{43210}.

16. 方程 $xyz=140000$ 有多少組正整數解？

 How many sets of positive integer solutions are there to the equation $xyz=140000$?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 設 a、x 爲實數，使得 $17x^4+34ax^3+(25a^2+10)x^2+(8a^3+10a)x+(a^4+25)=0$。求 a 所有可能值之積。

 Let a, x be real numbers such that $17x^4+34ax^3+(25a^2+10)x^2+(8a^3+10a)x+(a^4+25)=0$. Find the product of all possible values of a.

18. 有多少個四位正整數的數字之積是正平方數？

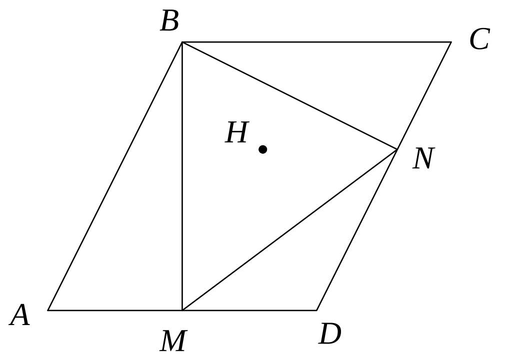
 How many four-digit positive integers have their product of digits being a positive square number?

19. 一個數學訓練課程從一月開始，為期十二個月。在每個月，學員均可選擇上課或自修。在單數月份（即一月、三月等），上課可得 0 分，自修可得 −1 分；在雙數月份（即二月、四月等），上課可得 1 分，自修可得 0 分。分數一直累積至課程結束。若規定在任何時候的累積分數都不能是負數，學員在整個課程的十二個月中有多少個上課或自修的組合可以選擇？

A mathematics training course starts in January and lasts for 12 months. In each month, a trainee may choose between attending classes or self-study. In odd-numbered months (i.e. Jan, Mar, etc.), a choice of attending classes is worth 0 mark while a choice of self-study is worth −1 mark. In even-numbered months (i.e. Feb, Apr, etc.), a choice of attending classes is worth 1 mark while a choice of self-study is worth 0 mark. The marks are accumulated towards the end of the course. If it is required that the accumulated marks at any time cannot be negative, how many different choices does a trainee have between attending classes and self-study in the 12 months?

20. 圖中，$ABCD$ 是平行四邊形，$\angle ABC$ 是鈍角。M 和 N 分別是 B 到 AD 和 CD 的垂足，而 H 則是 ΔBMN 的垂心。若 $MN = 3$ 而 $BD = 4$，求 BH。

In the figure, $ABCD$ is a parallelogram and $\angle ABC$ is obtuse. M and N are the feet of perpendiculars from B to AD and CD respectively, while H is the orthocentre of ΔBMN. If $MN = 3$ and $BD = 4$, find BH.



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