第十一屆培正數學邀請賽（**2012** 年）

11th Pui Ching Invitational Mathematics Competition (2012)

決賽（中三組）

Final Event (Secondary 3)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確値表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

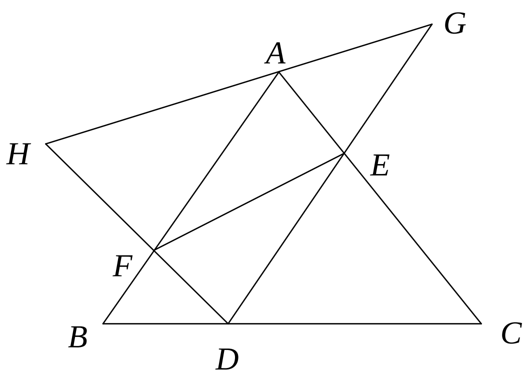
Questions 1 to 4 each carries 3 marks.

1. 現要安排五名編號爲 1 至 5 的學生圍著一張圓桌就坐。若兩名編號相差 1 的學生不可相鄰而坐，則安排座位的方法有多少種？（如果每人左方的人在兩種方法中皆相同，則這兩種安排座位的方法視爲相同。）

   Five students numbered 1 to 5 are seated at a round table. How many different seating arrangements are there if no two students whose numbers differ by 1 may sit next to each other? (Two seating arrangements are regarded to be the same if every person finds the same left-hand neighbour in the two arrangements.)

2. 圖中，*HAG*、*HFD*、*GED*、*AFB*、*AEC* 和 *BDC* 均爲直線，*HG* 與 *FE* 平行，且 ΔAFH 的面積爲 2012。若 *AFDE* 爲平行四邊形，求它的面積。

   In the figure, *HAG*, *HFD*, *GED*, *AFB*, *AEC* and *BDC* are straight lines, *HG* is parallel to *FE* and the area of ΔAFH is 2012. If *AFDE* is a parallelogram, find its area.



3. 求 $1^1+2^2+3^3+\cdots+1000^{1000}$ 的首四位數字。

   Find the first four digits of $1^1+2^2+3^3+\cdots+1000^{1000}$.

4. 在一個 3×3 方格表中，每格均有一個整數，使得每個橫行、每個直行和每條對角線上三個數之和均等於 *S*。附圖顯示了其中三格中的整數。求 *S* 的值。

   In a 3×3 table, there is an integer inside each cell so that the sum of the three numbers in each row, in each column and on each diagonal is equal to *S*. The figure shows the integers in three of the cells. Find the value of *S*.

12		
		5
		12

第 5 至第 8 題，每題 4 分。

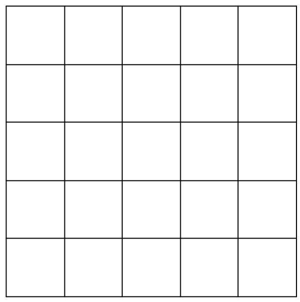
Questions 5 to 8 each carries 4 marks.

5. 一條自動行人道最初以固定速度從 A 向 B 運行，嘉希從 A 點沿自動行人道走往 B 點，共用了 30 秒，然後她沿自動行人道回到 A 點，再用了 60 秒。之後自動行人道的運行速度被調較至原來的兩倍，嘉希再次沿自動行人道向 B 點走，當她到達一半路程時，自動行人道突然停頓下來，然後嘉希繼續走向 B 點。假設嘉希的速度一直不變，則她第二次從 A 點走到 B 點共需多少秒？

   A moving walkway was initially set to move from A to B at a constant speed. Kathy walked from A along the moving walkway and took 30 seconds to reach B. She then walked back to A along the moving walkway in another 60 seconds. The moving walkway was then adjusted to move at twice its original speed. Kathy walked from A to B along the moving walkway again, and when she reached halfway the moving walkway suddenly stopped and then she continued to walk towards B. Assuming that Kathy's speed had remained constant throughout, how many seconds would Kathy take in her second journey from A to B?

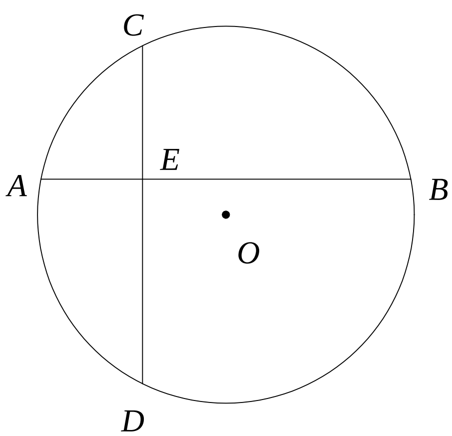
6. 有多少種方法可以從圖中的 5×5 方格表的 25 個小格中選 20 個小格，使得每橫行、每直行和兩條對角線上均各有 4 個小格被選？

   The figure shows a 5×5 grid with 25 small squares. In how many different ways can 20 small squares be chosen so that exactly 4 small squares are chosen in each row, each column and each of the two diagonals?



7. 圖中，O 是圓的中心，兩弦 AB 和 CD 相交於 E。若 AB 與 CD 互相垂直，$OE=5$，且圓半徑為 6，求 AB^2+CD^2。

   In the figure, O is the centre of the circle, and the two chords AB and CD meet at E. If AB is perpendicular to CD, $OE=5$ and the radius of the circle is 6, find AB^2+CD^2.



8. 當 2012 位數 444…44 除以 37 時，餘數是多少？

   What is the remainder when the 2012-digit number 444…44 is divided by 37?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 設 n 爲正整數，它有超過一個質因數，而各質因數之和爲 s。若 n 是 s 的倍數，求 n 的最小可能值。

   Let n be a positive integer with more than one prime factor. The sum of the prime factors of n is equal to s. If n is a multiple of s, find the smallest possible value of n.

10. 方程 $xyz = 1400$ 有多少組正整數解？

    How many sets of positive integer solutions are there to the equation $xyz = 1400$?

11. 已知 $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$，求 $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$。

    Given that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$, find $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$.

12. 老師在黑板上寫下 2012 個不同的整數，並要求學生從中選 3 個並加起來（同一個整數可以重複地選超過一次）。老師發現共有 k 個不同的答案。求 k 的最小可能值。

    The teacher wrote 2012 different integers on the board, and asked students to pick 3 integers and then add them up (the same integer can be picked more than once). The teacher found that there were altogether k different possible answers. Find the smallest possible value of k.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 若實數 x 滿足方程 $6x = 8^x$，求 x 所有可能值之積。

    If the real number x satisfies the equation $6x = 8^x$, find the product of all possible values of x.

14. 對於滿足 $0<x<1$ 的實數 x，記 $x=0.\overline{a_1a_2a_3\cdots}$（當 x 是截尾小數時，我們假設之後的小數位是 0，例如 $\frac{2}{5}=0.4000\ldots$），並設 $f(x)=0.\overline{b_1b_2b_3\cdots}$，其中 $b_i=\frac{a_i+a_{i+1}}{2}$ 並下捨入至最接近整數。例如： $f(0.74747474\ldots)=0.55555555\ldots=\frac{5}{9}$（我們把 $\frac{7+4}{2}=5.5$ 下捨入至 5）。現於黑板上寫上一個有理數 a，其中 $0<a<1$，然後一名學生每次會把黑板上的數 x 擦掉並寫上 $f(x)$ 取而代之，並重複此步驟直至黑板上的數一直保持不變爲止。若這個保持不變的數是 $\frac{m}{n}$，其中 n 是不超過 2012 的正整數，求 m 的最大可能值。

For a real number x satisfying $0<x<1$, denote $x=0.\overline{a_1a_2a_3\cdots}$ (when x is a terminating decimal, the subsequent decimal places are taken to be 0, e.g. $\frac{2}{5}=0.4000\ldots$), and let $f(x)=0.\overline{b_1b_2b_3\cdots}$, where $b_i=\frac{a_i+a_{i+1}}{2}$ rounded down to the nearest integer. For example, $f(0.74747474\ldots)=0.55555555\ldots=\frac{5}{9}$ ($\frac{7+4}{2}=5.5$ is rounded down to 5). Now the rational number a, where $0<a<1$, is written onto a blackboard. Each time a student will erase the number x on the blackboard and then write down the number $f(x)$ to replace it, and this process is repeated until the number on the blackboard remains constant. If this constant is $\frac{m}{n}$ where n is a positive integer not exceeding 2012, find the greatest possible value of m.

15. *ABCDEFGH* 是個正方體。一隻螞蟻可選其中一個頂點出發，沿正方體的棱爬到對角的頂點，當中必須途經所有頂點，而且不能經過同一點超過一次。那麼螞蟻爬行的路線有多少個不同的選擇？

ABCDEFGH is a cube. An ant may choose to start from one of the vertices and crawls along the edges of the cube to the opposite vertex, subject to the conditions that all vertices must be traversed and no point may be visited more than once. How many different choices are there for the route of the ant?

16. 方程 $4x^2-9y^2=2012^2$ 有多少組整數解？

How many sets of integer solutions are there to the equation $4x^2-9y^2=2012^2$?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

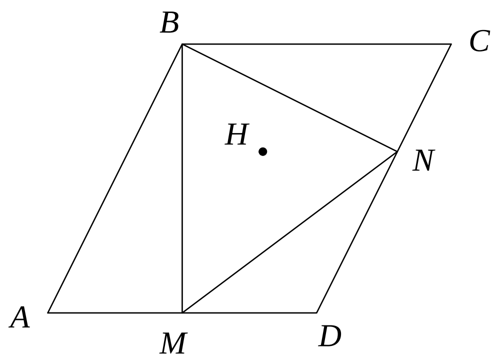
17. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字，且沒有字母代表的數字是 7。若 MATHS 所代表的五位數爲質數，求此質數。

In the multiplication shown, each letter represents a different digit from 0 to 9 and no letter represents 7. If the five-digit number represented by MATHS is prime, find this prime number.

$$\begin{array}{r} \mathrm{M\ A\ T\ H\ S} \\ \times \qquad\quad 7 \\ \hline \mathrm{P\ O\ I\ S\ O\ N} \end{array}$$

18. 圖中，*ABCD* 是平行四邊形，∠*ABC* 是鈍角。*M* 和 *N* 分別是 *B* 到 *AD* 和 *CD* 的垂足，而 *H* 則是 Δ*BMN* 的垂心。若 *MN* = 3 而 *BD* = 4，求 *BH*。

In the figure, *ABCD* is a parallelogram and ∠*ABC* is obtuse. *M* and *N* are the feet of perpendiculars from *B* to *AD* and *CD* respectively, while *H* is the orthocentre of Δ*BMN*. If *MN* = 3 and *BD* = 4, find *BH*.



19. 老師寫下了一個五位數 *N*，然後讓小陳看 *N* 的首三位數字、讓小李看 *N* 中間的三位數字、讓小張看 *N* 的最後三位數字。之後老師要求每人寫下一些他們知道關於 *N* 的性質。

小陳寫道：「*N* 不是 101 的倍數。」
小李寫道：「*N* 中間的三位數字之和是奇數。」
小張寫道：「*N* 不是 91 的倍數。」

求 *N* 的值。

The teacher wrote down a five-digit number *N* and then let Alan see the first three digits of *N*, let Bob see the three digits of *N* in the middle and let Carl see the last three digits of *N*. The teacher subsequently asked each person to write down something they knew about *N*.

Alan wrote, ‘*N* is not a multiple of 101’.
Bob wrote, ‘the sum of the three digits of *N* in the middle is odd.’
Carl wrote, ‘*N* is not a multiple of 91’.

Find the value of *N*.

20. 設 m 爲方程 $3x+2y+z=2008$ 正整數解的數目，n 爲方程 $3x+2y+z=2012$ 正整數解的數目。已知 m 的最後三位數字爲 002，求 n 的最後三位數字。

Let m be the number of positive integer solutions to the equation $3x+2y+z=2008$, and n be the number of positive integer solutions to the equation $3x+2y+z=2012$. Given that the last three digits of m are 002, find the last three digits of n.

全卷完

END OF PAPER