第十一屆培正數學邀請賽（**2012** 年）

11th Pui Ching Invitational Mathematics Competition (2012)

決賽（中二組）

Final Event (Secondary 2)

時限：**2** 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

注意：決賽的規則與初賽不同。除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Note: The rule in the Final Event is different from that in the Heat Event. Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 某校的 n 名學生從左至右排成一列。若從左邊數起的第 2012 名學生和從右邊數起的第 512 名學生是相鄰的兩人，求 n 所有可能值之和。

   The n students of a school are queued from left to right. If the 2012th student from the left and the 512th student from the right are adjacent to each other, find the sum of all possible values of n.

2. 某數學比賽設 20 道題，其中 3 分題、4 分題、5 分題、6 分題和 7 分題各佔 4 題。每題答對可得該題全部分數，否則該題得 0 分。若沛銓在比賽中得到 50 分，則他最多答對了幾題？

   There are 20 questions in a mathematical competition, with four of each of 3-mark, 4-mark, 5-mark, 6-mark and 7-mark questions. A correct answer merits all marks allocated to the question, and 0 mark is given otherwise. If Matthew gets 50 marks in the competition, what is the greatest number of questions which he answers correctly?

3. 一家百貨公司在加價 x% 後再提供 x% 的折扣優惠，結果貨品價格比之前便宜了 4%。求 x。

   A department store raised prices by x% and then provided a discount of x%. As a result, the actual prices were 4% lower than before. Find x.

4. 求 $1^1+2^2+3^3+\cdots+1000^{1000}$ 的首四位數字。

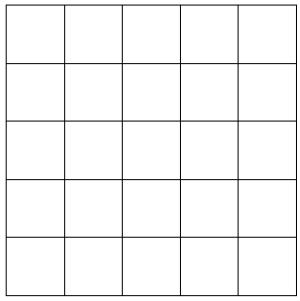
   Find the first four digits of $1^1+2^2+3^3+\cdots+1000^{1000}$.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5 有多少種方法可以從圖中的 5×5 方格表的 25 個小格中選 20 個小格，使得每橫行、每直行和兩條對角線上均各有 4 個小格被選？

The figure shows a 5×5 grid with 25 small squares. In how many different ways can 20 small squares be chosen so that exactly 4 small squares are chosen in each row, each column and each of the two diagonals?

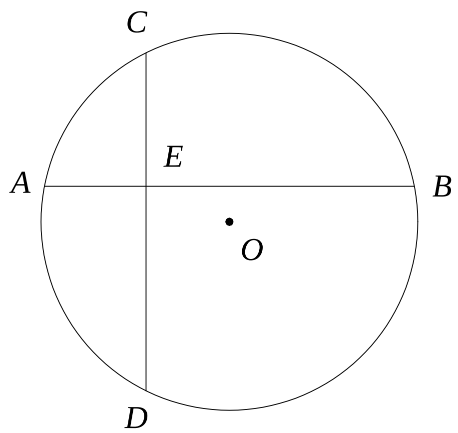


6. 某次數學競賽有一道填充題：「對任意的正整數 a、b，若 $ab+1$ 是 16 的倍數，則 $a+b$ 必定是 ____ 的倍數。」這題有很多不同的正確答案。求最大的一個正確答案。

In a mathematical competition there was a fill-in-the-blank question: ‘For any positive integers a and b, if $ab+1$ is a multiple of 16, then $a+b$ must be a multiple of ____.’ There are many correct answers to this question. Find the largest correct answer.

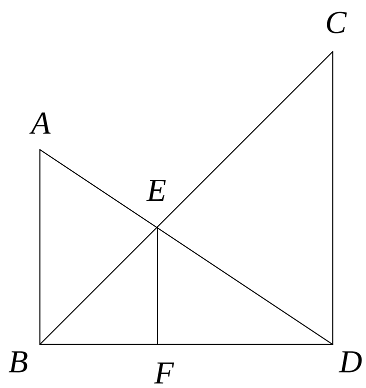
7. 圖中，O 是圓的中心，兩弦 AB 和 CD 相交於 E。若 AB 與 CD 互相垂直，$OE=5$，且圓半徑為 6，求 AB^2+CD^2。

In the figure, O is the centre of the circle, and the two chords AB and CD meet at E. If AB is perpendicular to CD, $OE=5$ and the radius of the circle is 6, find AB^2+CD^2.



8. 圖中，AED、BEC 和 BFD 均為直線，且 AB、CD 和 EF 均與 BD 垂直。若 $AB=7$、$EF=6$ 和 $CE=60$，求 BF。

In the figure, AED, BEC and BFD are straight lines, while AB, CD and EF are all perpendicular to BD. If $AB=7$, $EF=6$ and $CE=60$, find BF.



第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 某國家有 10 個城市。國王下令要興建 n 條道路，並須滿足以下條件。求 n 的最小可能值。

   - 每條道路連接兩個不同的城市。
   - 存在由 3 個城市組成的「觀光圈」（即可以從該 3 個城市的其中一個出發，經所建的道路到訪其餘 2 個城市，當中不會重複到訪城市也不會經過其他城市，最後返回起點）。
   - 存在由 4、5、…、10 個城市組成的「觀光圈」。

   There are 10 cities in a country. The king ordered n roads to be built subject to the following requirements. Find the smallest possible value of n.

   - Each road connects two different cities.
   - There exists a 'sightseeing loop' with 3 cities (which means one can start from one of the 3 cities and travel along the roads built to visit the other 2 cities without visiting the same city twice nor passing through other cities, and finally go back to the starting city).
   - There exists 'sightseeing loops' with 4, 5, …, 10 cities.

10. 現要安排 6 男 4 女排成一列而坐。若其中兩名男士為兄弟而必須相鄰而坐，且任何兩名女士都不能相鄰而坐，問共有多少種不同方法安排座位？

    6 men and 4 women are to be seated in a row. If two of the men are brothers and they must sit next to each other, and no two women may sit next to each other, how many different seating arrangements are there?

11. 已知 $p=a^2-b^2$，其中 p 是小於 100 的質數，而 a、b 則為整數。在 a 和 b 兩個整數中，其中一個是 2 的倍數，另一個是 3 的倍數。那麼 p 有多少個不同的可能值？

    Given $p=a^2-b^2$, where p is a prime number less than 100, while a and b are integers. Of the two integers a and b, one is divisible by 2 and the other is divisible by 3. How many different possible values of p are there?

12. 一個八位正整數 n 由數字 1、2、…、8 組成（沒有重複），且 n 可被 99 整除。求 n 的最大可能值。

    An eight-digit positive integer n is made up of the digits 1, 2, …, 8 without repetition, and is divisible by 99. What is the largest possible value of n?

第 13 至第 16 題，每題 6 分。

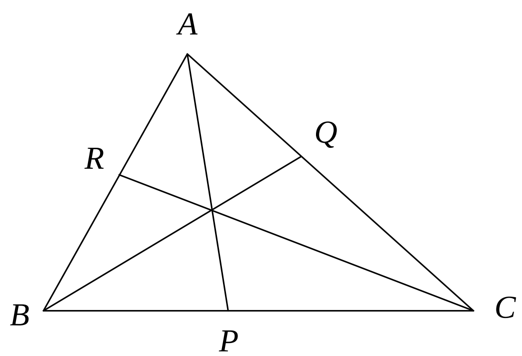
Questions 13 to 16 each carries 6 marks.

13. 方程 $4x^2-9y^2=2012^2$ 有多少組整數解？

How many sets of integer solutions are there to the equation $4x^2-9y^2=2012^2$?

14. 圖中，P 是 BC 的中點，Q、R 則分別是 CA 和 AB 上的點，使得 AP、BQ 和 CR 三線共點。若 $BP=21$、$BR=30$、$RA=15$ 且 $AQ=13$，求 ΔAQR 的面積。

In the figure, P is the mid-point of BC. Q and R are points on CA and AB respectively such that the lines AP, BQ and CR are concurrent. If $BP=21$, $BR=30$, $RA=15$ and $AQ=13$, find the area of ΔAQR.



15. 對於滿足 $0<x<1$ 的實數 x，記 $x=0.\overline{a_1a_2a_3\cdots}$（當 x 是截尾小數時，我們假設之後的小數位是 0，例如 $\frac{2}{5}=0.4000\ldots$），並設 $f(x)=0.\overline{b_1b_2b_3\cdots}$，其中 $b_i=\frac{a_i+a_{i+1}}{2}$ 並下捨入至最接近整數。例如：$f(0.74747474\ldots)=0.55555555\ldots=\frac{5}{9}$（我們把 $\frac{7+4}{2}=5.5$ 下捨入至 5）。現於黑板上寫上一個有理數 a，其中 $0<a<1$，然後一名學生每次會把黑板上的數 x 擦掉並寫上 $f(x)$ 取而代之，並重複此步驟直至黑板上的數一直保持不變爲止。若這個保持不變的數是 $\frac{m}{n}$，其中 n 是不超過 2012 的正整數，求 m 的最大可能值。

For a real number x satisfying $0<x<1$, denote $x=0.\overline{a_1a_2a_3\cdots}$ (when x is a terminating decimal, the subsequent decimal places are taken to be 0, e.g. $\frac{2}{5}=0.4000\ldots$), and let $f(x)=0.\overline{b_1b_2b_3\cdots}$, where $b_i=\frac{a_i+a_{i+1}}{2}$ rounded down to the nearest integer. For example, $f(0.74747474\ldots)=0.55555555\ldots=\frac{5}{9}$ ($\frac{7+4}{2}=5.5$ is rounded down to 5). Now the rational number a, where $0<a<1$, is written onto a blackboard. Each time a student will erase the number x on the blackboard and then write down the number $f(x)$ to replace it, and this process is repeated until the number on the blackboard remains constant. If this constant is $\frac{m}{n}$ where n is a positive integer not exceeding 2012, find the greatest possible value of m.

16. 方程 $xyz = 140000$ 有多少組正整數解？

How many sets of positive integer solutions are there to the equation $xyz = 140000$?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 老師寫下了一個五位數 N，然後讓小陳看 N 的首三位數字、讓小李看 N 中間的三位數字、讓小張看 N 的最後三位數字。之後老師要求每人寫下一些他們知道關於 N 的性質。

小陳寫道：「N 不是 101 的倍數。」
小李寫道：「N 中間的三位數字之和是奇數。」
小張寫道：「N 不是 91 的倍數。」

求 N 的值。

The teacher wrote down a five-digit number N and then let Alan see the first three digits of N, let Bob see the three digits of N in the middle and let Carl see the last three digits of N. The teacher subsequently asked each person to write down something they knew about N.

Alan wrote, 'N is not a multiple of 101'.
Bob wrote, 'the sum of the three digits of N in the middle is odd.'
Carl wrote, 'N is not a multiple of 91'.

Find the value of N.

18. 在所示的算式中，每個字母代表一個由 0 至 9 的不同數字，且沒有字母代表的數字是 7。若 MATHS 所代表的五位數為質數，求此質數。

In the multiplication shown, each letter represents a different digit from 0 to 9 and no letter represents 7. If the five-digit number represented by MATHS is prime, find this prime number.

$$\begin{array}{r} \text{M A T H S} \\ \times \qquad\quad 7 \\ \hline \text{P O I S O N} \end{array}$$

19. 設 $f(x) = 1 - |1 - 2x|$，且 $0 \le k \le 1$。若無窮數列 k、$f(k)$、$f(f(k))$、$f(f(f(k)))$、$\cdots$ 中剛好有 6 個不同的數，求 k 的最大可能值。

Let $f(x) = 1 - |1 - 2x|$ and $0 \le k \le 1$. If there are exactly 6 different numbers in the infinite sequence k, $f(k)$, $f(f(k))$, $f(f(f(k)))$, …, find the greatest possible value of k.

20. 一個數學訓練課程從一月開始，爲期十二個月。在每個月，學員均可選擇上課或自修。在單數月份（即一月、三月等），上課可得 0 分，自修可得 -1 分；在雙數月份（即二月、四月等），上課可得 1 分，自修可得 0 分。分數一直累積至課程結束。若規定在任何時候的累積分數都不能是負數，學員在整個課程的十二個月中有多少個上課或自修的組合可以選擇？

A mathematics training course starts in January and lasts for 12 months. In each month, a trainee may choose between attending classes or self-study. In odd-numbered months (i.e. Jan, Mar, etc.), a choice of attending classes is worth 0 mark while a choice of self-study is worth -1 mark. In even-numbered months (i.e. Feb, Apr, etc.), a choice of attending classes is worth 1 mark while a choice of self-study is worth 0 mark. The marks are accumulated towards the end of the course. If it is required that the accumulated marks at any time cannot be negative, how many different choices does a trainee have between attending classes and self-study in the 12 months?

全卷完

END OF PAPER