第九屆培正數學邀請賽

9th Pui Ching Invitational Mathematics Competition

決賽（高中組）

Final Event (Senior Secondary)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 設 $[x]$ 代表不超過 x 的最大整數，且 $\{x\}=x-[x]$，例如 [1.1] = 1、{6.9} = 0.9 和 [5] = 5。若 $2^x=40$，求 $2^{\{x\}}$ 的值。

   Let $[x]$ denote the greatest integer not exceeding x and $\{x\}=x-[x]$. For example, [1.1] = 1, {6.9} = 0.9 and [5] = 5. If $2^x=40$, find the value of $2^{\{x\}}$.

2. 小蔚把 20100320 的數字重新排列，得到一個可被 11 整除的八位數。求此八位數的最小可能值。

   By permuting the digits of 20100320, Judy obtained an eight-digit number divisible by 11. Find the smallest possible value of this eight-digit number.

3. 對於正整數 n，設 $n!=1\times2\times3\times\cdots\times n$，例如：$4!=1\times2\times3\times4=24$。求最小的正整數 k，使得 k 的數字之積等於 6!。

   For positive integer n, let $n!=1\times2\times3\times\cdots\times n$. For instance, $4!=1\times2\times3\times4=24$. Find the smallest positive integer k for which the product of the digits of k equals 6!.

4. 已知對任意正整數 n，$1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$。

   求 $1^2+4^2+7^2+\cdots+31^2$ 的值。

   It is given that $1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for any positive integer n.

   Find the value of $1^2+4^2+7^2+\cdots+31^2$.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 若 $x_1=2010$，且對任意正整數 n 皆有 $\dfrac{1}{x_n}-\dfrac{1}{x_nx_{n+1}}=1$，求 $x_1x_2\cdots x_{100}$ 的值。

   If $x_1=2010$ and $\dfrac{1}{x_n}-\dfrac{1}{x_nx_{n+1}}=1$ for all positive integers n, find the value of $x_1x_2\cdots x_{100}$.

6. 已知 $f(x)$ 爲二次函數，於 $x=m$ 時達至極大值。若 $\int_2^7 f(x)\mathrm{d}x=\int_3^8 f(x)\mathrm{d}x$ ，求 m 的值。

Let $f(x)$ be a quadratic function which attains its maximum value at $x=m$. If $\int_2^7 f(x)\mathrm{d}x=\int_3^8 f(x)\mathrm{d}x$, find the value of m.

7. 現有 10 種不同顏色的咭片，每種顏色各 3 張，並分別寫上「1」、「2」和「3」。現要選出 3 張不同的咭片，使得其中兩張咭片上的數字之和等於第三張咭片上的數字，問共有多少個不同的選擇？

There are cards in 10 different colours. For each colour, there are 3 cards, with ‘1’, ‘2’ and ‘3’ written respectively. Now 3 cards are to be chosen so that the sum of the numbers on two of the cards is equal to the number on the third card. How many different choices are there?

8. 浩恩有 9 張分別寫上 1 至 9 的咭片。現在他要利用這 9 張咭片組成 3 個能被 9 整除的三位數並把它們從小至大排列。他有多少種不同的方法組成這些三位數？

Fred has 9 cardboards with the numbers 1 to 9 written. He has to form 3 three-digit numbers which are divisible by 9 with these cardboards and arrange them in ascending order. In how many different ways can he form these three-digit numbers?

第 9 至第 12 題，每題 5 分。

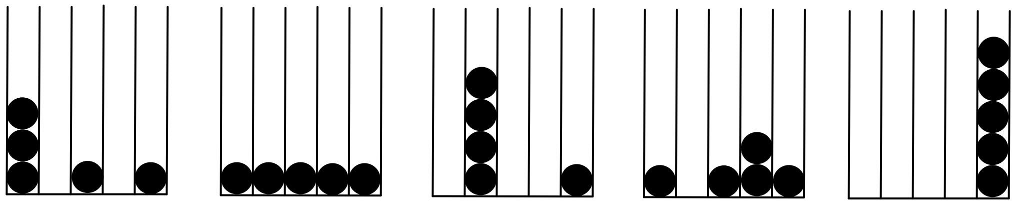
Questions 9 to 12 each carries 5 marks.

9. 子揚想了四個正整數 a、b、c 和 d，其中 $a>b>c>d$ 。他求得 $a-b$ 、$a-c$ 、$a-d$ 、$b-c$ 、$b-d$ 和 $c-d$ 的值後，發現其中四個是 3、7、10 和 20。求 $a-d$ 的最大可能值。

Henry thought of four positive integers a, b, c and d where $a>b>c>d$. After computing the values of $a-b$, $a-c$, $a-d$, $b-c$, $b-d$ and $c-d$, he found that four of these values were 3, 7, 10 and 20. Find the greatest possible value of $a-d$.

10. 在一個遊戲中，參加者需把 5 個球射進 5 條垂直的坑道。如果 5 個球最終成一直線，參加者便勝出。（下圖顯示了 5 個例子，當中只有第二個和第五個勝出。）若每個球都必定會射進其中一條坑道，且射進各坑道的機會均等，求勝出的概率。

In a game, the player has to shoot 5 balls into one of the 5 vertical tracks, and wins if the 5 balls turn out to be collinear. (The figure below shows 5 examples, among which only the second and fifth outcomes are winning.) If each ball must get into one of the tracks with equal probability, find the probability of winning.



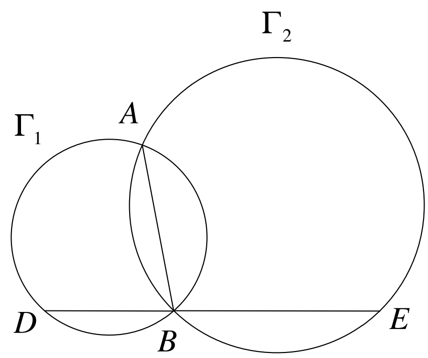
11. 在所示的乘式中，每個字母代表一個由 0 至 9 的不同數字。求乘積 ADABBC 所代表的六位數。

In the multiplication shown, each letter represents a different digit from 0 to 9. Find the six-digit number represented by the product ADABBC.

$$\begin{array}{r} \text{A B C} \\ \times \quad \text{C B A} \\ \hline \text{A D A B B C} \end{array}$$

12. 圖中，兩圓 Γ_1 和 Γ_2 交於 A 和 B。一條穿過 B 的直線交 Γ_1 於 D，交 Γ_2 於 E。若 AD 切圓 Γ_2 於 A，AE 切圓 Γ_1 於 A，$AB=3$，$AD=6$ 且 ΔADE 的面積爲 $\sqrt{k}$，求 k。

In the figure, the circles Γ_1 and Γ_2 meet at A and B. A line through B meets Γ_1 at D and Γ_2 at E. If AD is tangent to Γ_2 at A, AE is tangent to Γ_1 at A, $AB=3$, $AD=6$ and the area of ΔADE is $\sqrt{k}$, find k.



第 13 至第 16 題，每題 6 分。

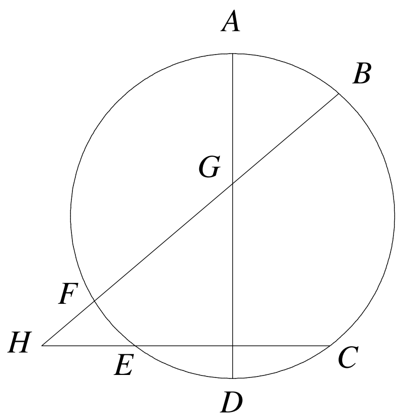
Questions 13 to 16 each carries 6 marks.

13. 若 $f(x)=\sin[(x^2-x)\pi]$，則方程 $f'(x)=0$ 在 $0\le x\le 10$ 區間內有多少個解？

If $f(x)=\sin[(x^2-x)\pi]$, how many solutions are there to the equation $f'(x)=0$ in the interval $0\le x\le 10$?

14. 圖中，AD 是圓的直徑，B、C、E、F 爲圓周上的四點，其中 BF 交 AD 於 G，且 BF 與 CE 延長後交於 H。若 $\angle FEH = 54°$、$\angle BGD = 126°$、$\angle CED = 36°$ 且 $\angle BEC = x°$，求 x。

In the figure, AD is a diameter of the circle. B, C, E, F are points on the circumference such that BF meets AD at G and that the extensions of BF and CE meet at H. If $\angle FEH = 54°$, $\angle BGD = 126°$, $\angle CED = 36°$ and $\angle BEC = x°$, find x.



15. 設 a 爲常數。若不等式 $\left|x^2-2ax-a^2-\frac{3}{4}\right|\le 1$ 對任意 $0\le x\le 1$ 成立，求 a^2 的最大可能值。

Let a be a constant. If the inequality $\left|x^2-2ax-a^2-\frac{3}{4}\right|\le 1$ holds for all $0\le x\le 1$, find the greatest possible value of a^2.

16. 在 ΔABC 中，$AB=AC=x+1$ 且 $BC=1$。已知平面上滿足 $DA=DC$ 和 $DB=BC$ 的 D 點只有一個，求 x 的值。

In ΔABC, $AB=AC=x+1$ and $BC=1$. Given that there is only one point D on the plane satisfying $DA=DC$ and $DB=BC$, find the value of x.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在坐標平面上，R 是以 (0, 0)、(8, 0)、(8, 6) 和 (0, 6) 爲頂點的長方形，而 S 則是以 (5, 1)、(7, 1)、(7, 3) 和 (5, 3) 爲頂點的正方形。一條斜率爲 m 的直線把 R 分成面積相同的兩部分，同時把 S 分成面積比爲 $1:7$ 的兩部分。求 m 所有可能值之和。

On the coordinate plane, R is the rectangle with vertices (0, 0), (8, 0), (8, 6) and (0, 6) while S is the square with vertices (5, 1), (7, 1), (7, 3) and (5, 3). A straight line with slope m divides R into two parts of the same area, and at the same time divides S into two parts whose areas are in ratio $1:7$. Find the sum of all possible values of m.

18. 在 6 張不同顏色的咭紙上分別印有式子 $x^2 - \square\, x + \square = 0$ ，即共有 12 個空格。現有 12 張貼紙，分別印上 1、2、⋯、12，並容許在每個空格貼上一張貼紙，使得 6 條二次方程的所有根均為整數。問共有多少種不同的方法貼上貼紙？

The expression $x^2 - \square\, x + \square = 0$ is printed on 6 cardboards of different colours, so that there are altogether 12 blanks. One is given 12 stickers, with the numbers 1, 2, …, 12 printed on them, and is allowed to put one sticker onto each blank, so that all roots of the 6 resulting quadratic equations are integers. In how many different ways can the stickers be put onto the blanks?

19. 求 $\int_0^1 \frac{x^2+2}{(x^2+1)^3}\,\mathrm{d}x$ 的值。

Evaluate $\int_0^1 \frac{x^2+2}{(x^2+1)^3}\,\mathrm{d}x$.

20. 已知 a_1 、 a_2 、⋯、 a_{10} 是非負實數。在以下方程組中求 a_{10} 。

Given a_1, a_2, …, a_{10} are non-negative real numbers. Find a_{10} in the following system of equations.

$$\begin{cases} a_1 + a_2 + a_3 + \cdots + a_{10} = 104 \\ a_1 + 2a_2 + 3a_3 + \cdots + 10a_{10} = 740 \\ a_1 + 4a_2 + 9a_3 + \cdots + 100a_{10} = 5300 \\ a_1 + 8a_2 + 27a_3 + \cdots + 1000a_{10} = 38300 \end{cases}$$

全卷完

END OF PAPER