第九屆培正數學邀請賽

9th Pui Ching Invitational Mathematics Competition

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 一家百貨店進行「買三送一」推廣，顧客每次購買三件貨品，可獲贈一件貨品，但贈品的價值須分別不高於購買的三件貨品的價值。美詩希望得到 15 件價值分別是 1 元、2 元、…、15 元的貨品，則她最少要付多少元？

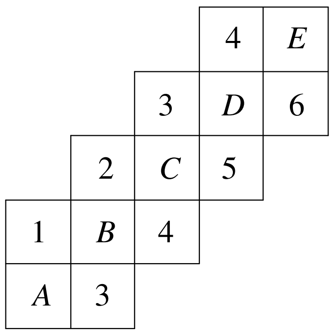
   A department store has launched a ‘buy 3 get 1 free’ promotion. Each time when one buys 3 articles, one is entitled to get an article for free, subject to the condition that the value of the free article must not exceed each of the three purchased articles. If Macy wants to get 15 articles valued at 1 dollar, 2 dollars, …, 15 dollars respectively, what is the minimum amount (in dollars) that she has to pay?

2. 設 $[x]$ 代表不超過 x 的最大整數，且 $\{x\}=x-[x]$，例如 $[1.1]=1$、$\{6.9\}=0.9$ 和 $[5]=5$。若 $2^x=40$，求 $2^{\{x\}}$ 的值。

   Let $[x]$ denote the greatest integer not exceeding x and $\{x\}=x-[x]$. For example, $[1.1]=1$, $\{6.9\}=0.9$ and $[5]=5$. If $2^x=40$, find the value of $2^{\{x\}}$.

3. 附圖是一個由 13 個 1×1 方格組成的圖案，其中每個方格中都有一個正整數。若當中任何一個 2×2 正方形中的四個方格內的各數之和均是相同的，求 $A+B$ 的最小可能值。

   The figure shown is made up of 13 grids of size 1×1, and inside each grid there is a positive integer. If the sum of the numbers in the four grids of any 2×2 square is the same, find the smallest possible value of $A+B$.



4. 小蔚把 20100320 的數字重新排列，得到一個可被 11 整除的八位數。求此八位數的最小可能值。

   By permuting the digits of 20100320, Judy obtained an eight-digit number divisible by 11. Find the smallest possible value of this eight-digit number.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 已知 $f(x)$ 爲二次函數，於 $x=m$ 時達至極大值 n，其中 $n<0$。已知拋物線 $y=f(x)$ 和直線 $y=0$、$x=2$、$x=7$ 所圍出的面積等於 $y=f(x)$ 和直線 $y=0$、$x=3$、$x=8$ 所圍出的面積，求 m 的值。

   Let $f(x)$ be a quadratic function which attains its maximum value n at $x=m$, where $n<0$. Given that the area bounded by the parabola $y=f(x)$ and the straight lines $y=0$, $x=2$ and $x=7$ is equal to the area bounded by $y=f(x)$ and the straight lines $y=0$, $x=3$ and $x=8$, find the value of m.

6. 在所示的乘式中，每個字母代表一個由 0 至 9 的不同數字。求乘數 PQPP 所代表的四位數。

   In the multiplication shown, each letter represents a different digit from 0 to 9. Find the four-digit number represented by the multiplier PQPP.

$$\begin{array}{r} \text{P Q P P P} \\ \times \quad \text{P Q P P} \\ \hline \text{R P P P P P P R P} \end{array}$$

7. 小燊有 10 張分別寫上 0 至 9 的咭片。現在他要利用這 10 張咭片組成 5 個能被 9 整除的兩位數並把它們從小至大排列。他有多少種不同的方法組成這些兩位數？

   Eric has 10 cardboards with the numbers 0 to 9 written. He has to form 5 two-digit numbers which are divisible by 9 with these cardboards and arrange them in ascending order. In how many different ways can he form these two-digit numbers?

8. 在四邊形 $ABCD$ 中，$AB=BC$、$\angle ABC=\angle CDA=90°$ 且 $CD+DA=10$。求 $ABCD$ 的面積。

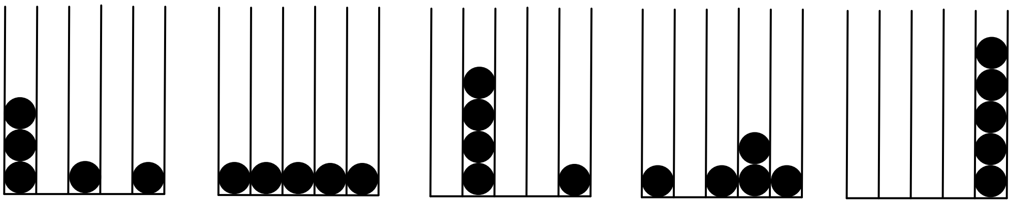
   In quadrilateral $ABCD$, $AB=BC$, $\angle ABC=\angle CDA=90°$ and $CD+DA=10$. Find the area of $ABCD$.

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 在一個遊戲中，參加者需把 5 個球射進 5 條垂直的坑道。如果 5 個球最終成一直線，參加者便勝出。（下圖顯示了 5 個例子，當中只有第二個和第五個勝出。）若每個球都必定會射進其中一條坑道，且射進各坑道的機會均等，求勝出的概率。

   In a game, the player has to shoot 5 balls into one of the 5 vertical tracks, and wins if the 5 balls turn out to be collinear. (The figure below shows 5 examples, among which only the second and fifth outcomes are winning.) If each ball must get into one of the tracks with equal probability, find the probability of winning.



10. 設 a、b、c 爲滿足 $a+b+c=1$ 的非負實數。求 $\frac{1+2ab+2bc+2ca}{a+2b+3c}$ 的最小可能值。

    Let a, b, c be non-negative real numbers satisfying $a+b+c=1$. Find the smallest possible value of $\frac{1+2ab+2bc+2ca}{a+2b+3c}$.

11. 方程 $\cos[(x^2+20x+10)\pi]=0$ 在 $0\le x\le 10$ 區間內有多少個解？

    How many solutions are there to the equation $\cos[(x^2+20x+10)\pi]=0$ in the interval $0\le x\le 10$?

12. 設 $[x]$ 代表不超過 x 的最大整數，且 $\{x\}=x-[x]$，例如 [1.1] = 1、{6.9} = 0.9 和 [5] = 5。若 x 是正數，且 $2[x]\{x\}+4=3\{x\}+3x$ ，求 x 所有可能值之和。

    Let $[x]$ denote the greatest integer not exceeding x and $\{x\}=x-[x]$. For example, [1.1] = 1, {6.9} = 0.9 and [5] = 5. If x is a positive number such that $2[x]\{x\}+4=3\{x\}+3x$, find the sum of all possible values of x.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 嘉美想了四個正整數 a、b、c 和 d。她求得 $a+b$、$a+c$、$a+d$、$b+c$、$b+d$ 和 $c+d$ 的值後，發現其中四個是 41、70、82 和 98。求 $a+b+c+d$ 的值。

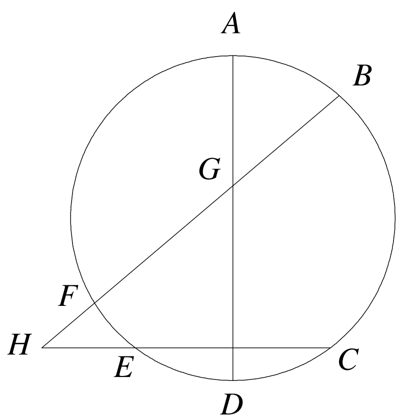
    Gloria thought of four positive integers a, b, c and d. After computing the values of $a+b$, $a+c$, $a+d$, $b+c$, $b+d$ and $c+d$, she found that four of these values were 41, 70, 82 and 98. Find the value of $a+b+c+d$.

14. 若要選兩個正整數使得它們的最大公因數是 5 而最小公倍數是 900，有多少個不同的選擇？

    If two positive integers are to be chosen such that their H.C.F. is 5 and their L.C.M. is 900, how many different choices are there?

15. 圖中，AD 是圓的直徑，B、C、E、F 為圓周上的四點，其中 BF 交 AD 於 G，且 BF 與 CE 延長後交於 H。若 $\angle FEH = 54°$、$\angle BGD = 126°$、$\angle CED = 36°$ 且 $\angle BEC = x°$，求 x。

    In the figure, AD is a diameter of the circle. B, C, E, F are points on the circumference such that BF meets AD at G and that the extensions of BF and CE meet at H. If $\angle FEH = 54°$, $\angle BGD = 126°$, $\angle CED = 36°$ and $\angle BEC = x°$, find x.



16. 在 ΔABC 中，$AB = AC = x+1$ 且 $BC = 1$。已知平面上滿足 $DA = DC$ 和 $DB = BC$ 的 D 點只有一個，求 x 的值。

    In ΔABC, $AB = AC = x+1$ and $BC = 1$. Given that there is only one point D on the plane satisfying $DA = DC$ and $DB = BC$, find the value of x.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在坐標平面上，R 是以 (0, 0)、(8, 0)、(8, 6) 和 (0, 6) 爲頂點的長方形，而 S 則是以 (5, 1)、(7, 1)、(7, 3) 和 (5, 3) 爲頂點的正方形。一條斜率爲 m 的直線把 R 分成面積相同的兩部分，同時把 S 分成面積比爲 $1:7$ 的兩部分。求 m 所有可能值之和。

    On the coordinate plane, R is the rectangle with vertices (0, 0), (8, 0), (8, 6) and (0, 6) while S is the square with vertices (5, 1), (7, 1), (7, 3) and (5, 3). A straight line with slope m divides R into two parts of the same area, and at the same time divides S into two parts whose areas are in ratio $1:7$. Find the sum of all possible values of m.

18. 在 6 張不同顏色的咭紙上分別印有式子 $x^2-\square x+\square=0$ ，即共有 12 個空格。現有 12 張貼紙，分別印上 1、2、⋯、12，並容許在每個空格貼上一張貼紙，使得 6 條二次方程的所有根均爲整數。問共有多少種不同的方法貼上貼紙？

    The expression $x^2-\square x+\square=0$ is printed on 6 cardboards of different colours, so that there are altogether 12 blanks. One is given 12 stickers, with the numbers 1, 2, …, 12 printed on them, and is allowed to put one sticker onto each blank, so that all roots of the 6 resulting quadratic equations are integers. In how many different ways can the stickers be put onto the blanks?

19. 在坐標平面上，一個機械人位於點 (0, 0)。現有一個可操控機械人的遙控器，上面有四個按鈕，分別對應「右移 2 單位」、「右移 3 單位」、「上移 1 單位」和「上移 2 單位」。若要使用這些按鈕使機械人移到點 (7, 5)，則按鈕的次序有多少個不同的可能性？

    On the coordinate plane, there is a robot at the point (0, 0). There is a remote control for the robot with four buttons, corresponding to the instructions ‘move 2 units to the right’, ‘move 3 units to the right’, ‘move 1 unit upward’ and ‘move 2 units upward’. Now the robot is to travel to the point (7, 5) by pressing these buttons. How many different sequences of pressing the buttons are possible?

20. 已知 a_1 、 a_2 、⋯、 a_{10} 是非負實數。在以下方程組中求 a_{10} 。

Given a_1 , a_2 , …, a_{10} are non-negative real numbers. Find a_{10} in the following system of equations.

$$\begin{cases} a_1 + a_2 + a_3 + \cdots + a_{10} = 104 \\ a_1 + 2a_2 + 3a_3 + \cdots + 10a_{10} = 740 \\ a_1 + 4a_2 + 9a_3 + \cdots + 100a_{10} = 5300 \\ a_1 + 8a_2 + 27a_3 + \cdots + 1000a_{10} = 38300 \end{cases}$$

全卷完

END OF PAPER