第九屆培正數學邀請賽

9th Pui Ching Invitational Mathematics Competition

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分為 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均為十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

1. 圖中所示的是一個周界為 8 的軸對稱圖形，它由三個邊長為 1 的正方形拼合而成。若 X 和 Y 是周界上的兩點，且 X 和 Y 的直線距離是 d，求 d 的最大可能值。

    The figure shown has perimeter 8, possesses reflectional symmetry and is made up of 3 squares of side length 1. If X and Y are two points on the perimeter and the straight-line distance between X and Y is d, find the greatest possible value of d.
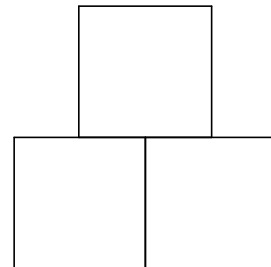
2. 一家百貨店進行「買三送一」推廣，顧客每次購買三件貨品，可獲贈一件貨品，但贈品的價值須分別不高於購買的三件貨品的價值。美詩希望得到 15 件價值分別是 1 元、2 元、…、15 元的貨品，則她最少要付多少元？

    A department store has launched a 'buy 3 get 1 free' promotion. Each time when one buys 3 articles, one is entitled to get an article for free, subject to the condition that the value of the free article must not exceed each of the three purchased articles. If Macy wants to get 15 articles valued at 1 dollar, 2 dollars, …, 15 dollars respectively, what is the minimum amount (in dollars) that she has to pay?

3. 附圖是一個由 13 個 1×1 方格組成的圖案，其中每個方格中都有一個正整數。若當中任何一個 2×2 正方形中的四個方格內的各數之和均是相同的，求 $A+B$ 的最小可能值。

    The figure shown is made up of 13 grids of size 1×1, and inside each grid there is a positive integer. If the sum of the numbers in the four grids of any 2×2 square is the same, find the smallest possible value of $A+B$.
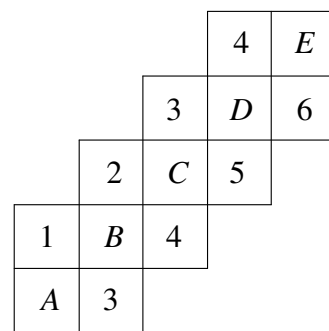
4. 對於正整數 n，設 $n!=1\times2\times3\times\cdots\times n$，例如：$4!=1\times2\times3\times4=24$。求最小的正整數 k，使得 k 的數字之積等於 6!。

    For positive integer n, let $n!=1\times2\times3\times\cdots\times n$. For instance, $4!=1\times2\times3\times4=24$. Find the smallest positive integer k for which the product of the digits of k equals 6!.

5. 詩雅有兩張全等的長方形紙。她用兩種不同的方法將它們拼成一個大長方形。這兩個大長方形的對角線的長度分別是 19 和 22。求長方形紙的對角線的長度。

    Alice had two congruent sheets of paper in rectangular shape. She built a big rectangle with them by two different methods. It is known that the lengths of the diagonals of two big rectangles are 19 and 22. Find the length of the diagonal of the rectangular sheet of paper.

6. 在所示的乘式中，每個字母代表一個由 0 至 9 的不同數字。求乘積 BCADEEE 所代表的七位數。

    In the multiplication shown, each letter represents a different digit from 0 to 9. Find the seven-digit number represented by the product BCADEEE.

    $$\begin{array}{r} A\,A\,B\,A \\ \times \quad A\,B\,B\,A \\ \hline B\,C\,A\,D\,E\,E\,E \end{array}$$

7. 某班有 9 名學生，學號分別是 1 至 9。若要把他們分成 3 組，每組 3 人，使得每組的學生的學號之和都是 9 的倍數，共有多少種分組方法？

    In a class there are 9 students, numbered 1 to 9. In how many different ways can the students be divided into 3 groups of 3 students each such that the sum of the numbers of the students in each group is divisible by 9?

8. 現有 10 種不同顏色的咭片，每種顏色各 3 張，並分別寫上「1」、「2」和「3」。現要選出 3 張不同的咭片，使得其中兩張咭片上的數字之和等於第三張咭片上的數字，問共有多少個不同的選擇？

    There are cards in 10 different colours. For each colour, there are 3 cards, with '1', '2' and '3' written respectively. Now 3 cards are to be chosen so that the sum of the numbers on two of the cards is equal to the number on the third card. How many different choices are there?

9. 圖中，*ABC* 是銳角三角形，*D* 是 *A* 到 *BC* 的垂足。若 $AD = 8$、$BC = 11$ 且 $AB^2 - AC^2 = 66$，求 ΔACD 的面積。

   In the figure, *ABC* is an acute-angled triangle. *D* is the foot of perpendicular from *A* to *BC*. If $AD = 8$, $BC = 11$ and $AB^2 - AC^2 = 66$, find the area of ΔACD.
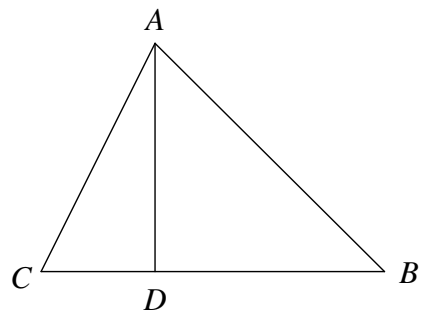
10. 兩名運動員在圓形的跑步徑上分別以均速向相同的方向跑步。他們同時在 *O* 點出發，之後不斷在跑步徑上相遇。設他們第 1 次、第 2 次和第 5 次相遇的地方分別爲 *A*、*B* 和 *E*。若沿跑步徑量度，*O* 和 *A* 的最短距離是 100 米，而 *E* 和 *B* 的最短距離則是 230 米。求跑步徑的長度（以米爲單位）。

    Two athletes ran on a circular jogging track in the same direction, each at uniform speed. They started from point *O* at the same time and then met on the track repeatedly. Let their first, second and fifth meeting points be *A*, *B* and *E* respectively. When measured along the track, the shortest distance between *O* and *A* is 100 m while the shortest distance between *E* and *B* is 230 m. Find the length of the track (in m).

11. 設 *a*、*b*、*c* 爲滿足 $a + b + c = 1$ 的非負實數。求 $\dfrac{1 + 2ab + 2bc + 2ca}{a + 2b + 3c}$ 的最小可能值。

    Let *a*, *b*, *c* be non-negative real numbers satisfying $a + b + c = 1$. Find the smallest possible value of $\dfrac{1 + 2ab + 2bc + 2ca}{a + 2b + 3c}$.

12. 設 [*x*] 代表不超過 *x* 的最大整數，例如 [1.1] = 1、[6.9] = 6 和 [5] = 5。若 $\left[0.1n\pi\right] = \left[0.1(n-1)\pi\right]$，其中 *n* 是不超過 100 的正整數，問 *n* 有多少個不同的可能值？

    Let [*x*] denote the greatest integer not exceeding *x*. For example, [1.1] = 1, [6.9] = 6 and [5] = 5. If $\left[0.1n\pi\right] = \left[0.1(n-1)\pi\right]$ where *n* is a positive integer not exceeding 100, how many different possible values of *n* are there?

13. 設 $g(x) = \dfrac{x}{4} + 4 + \sqrt{x-4}$ 。求 $\underbrace{g(g(g(\cdots g}_{10 \text{ 個 } g}(13)\cdots)))$ 的值，答案準確至小數點後 3 位。

    Let $g(x) = \dfrac{x}{4} + 4 + \sqrt{x-4}$ 。Find the value of $\underbrace{g(g(g(\cdots g}_{10 \text{ copies of } g}(13)\cdots)))$ correct to 3 decimal places.

14. 若要選兩個正整數使得它們的最大公因數是 5 而最小公倍數是 900，有多少個不同的選擇？

    If two positive integers are to be chosen such that their H.C.F. is 5 and their L.C.M. is 900, how many different choices are there?

15. 某班同學進行考試，試卷共設 5 題，題號分別是 1 至 5。對任意兩題，同時答對這兩題的人數都剛好是該兩題的題號之和（例如有 $1+3=4$ 名同學同時答對第 1 題和第 3 題，有 $2+5=7$ 名同學同時答對第 2 題和第 5 題，如此類推）。問該班的學生最少有幾人？

    A class of students sat for an examination. There were 5 questions, numbered 1 to 5. For any two questions, the number of students who answered both of them correctly is equal to the sum of the two question numbers. (For example, $1+3=4$ students answered both Questions 1 and 3 correctly, $2+5=7$ students answered both Questions 2 and 5 correctly, and so on.) What is the least number of students in the class?

16. 美芬想了四個正整數 a、b、c 和 d，其中 $a>b>c>d$ 。她求得 $a-b$、$a-c$、$a-d$、$b-c$、$b-d$ 和 $c-d$ 的值後，發現其中四個是 7、9、14 和 16。求 $a-d$ 所有可能值之和。

    Ivy thought of four positive integers a, b, c and d where $a>b>c>d$. After computing the values of $a-b$, $a-c$, $a-d$, $b-c$, $b-d$ and $c-d$, she found that four of these values were 7, 9, 14 and 16. Find the sum of all possible values of $a-d$.

17. 設 x、y、z 為正整數，其中 $x < y < z$ 且 $x+y+z=155$。敏儀計算了 x 和 y 的最大公因數、x 和 z 的最大公因數及 y 和 z 的最大公因數，並發現這三個最大公因數當中，其中一個等於另外兩個之積。求 x 所有可能值之和。

Let x, y, z be positive integers such that $x < y < z$ and $x+y+z=155$. Mandy computes the H.C.F. of x and y, the H.C.F. of x and z as well as the H.C.F. of y and z. She finds that among these three H.C.F.s, one is equal to the product of the other two. Find the sum of all possible values of x.

18. 圖中，ABC 是等腰三角形，其中 $AB = AC$。四個圓之間任何連續的兩個均互相外切，且每個均與 AB 和 AC 相切，最大的一個圓同時與 BC 相切。當中最小和最大圓的半徑分別是 8 和 27。已知 ΔABC 的面積可寫成 $p\sqrt{q}$，其中 p、q 為正整數，求 p 的最大可能值。

In the figure, ΔABC is isosceles with $AB = AC$. The four circles are such that any two consecutive circles are externally tangent to each other, each is tangent to AB and AC, and the largest one is tangent to BC as well. The radii of the smallest and largest circles are 8 and 27 respectively. Given that the area of ΔABC can be expressed as $p\sqrt{q}$ where p, q are positive integers, find the greatest possible value of p.
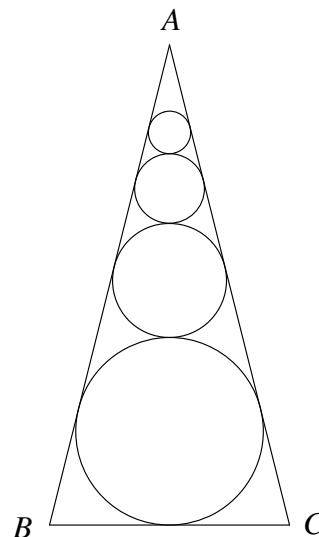
19. 在坐標平面上，一個機械人位於點 $(0, 0)$。現有一個可操控機械人的遙控器，上面有四個按鈕，分別對應「右移 2 單位」、「右移 3 單位」、「上移 1 單位」和「上移 2 單位」。若要使用這些按鈕使機械人移到點 $(7, 5)$，則按鈕的次序有多少個不同的可能性？

On the coordinate plane, there is a robot at the point $(0, 0)$. There is a remote control for the robot with four buttons, corresponding to the instructions 'move 2 units to the right', 'move 3 units to the right', 'move 1 unit upward' and 'move 2 units upward'. Now the robot is to travel to the point $(7, 5)$ by pressing these buttons. How many different sequences of pressing the buttons are possible?

20. 在課室裡，老師對五名學生小陳、小李、小張、小王和小何說：「我寫下了一個五位數 N，它由五個不同的數字組成。我會讓小陳看 N 的萬位和千位，讓小李看千位和百位，讓小張看百位和十位，讓小王看十位和個位，並讓小何看個位和萬位。」之後老師如所述般讓每名學生知道 N 的兩個數字，然後各人圍圈而坐，展開了以下的對話。

老師說：「知道 N 的一個質因數的請舉手」，之後有兩名學生舉手。

老師再問：「知道 N 的一個質因數的請舉手」，這次有三名學生舉手。

老師說：「知道 N 的一個合成數因數的請舉手」，之後有兩名學生舉手。

老師又說：「知道 N 的兩個合成數因數的請舉手」，但沒有學生舉手。

之後老師問：「誰知道 N 的值？」

其中一名學生說：「我知道，N 是 59 的倍數呢。」

假設所有學生都是聰明的（即有足夠資料便能作出推論），求 N。

In a classroom, the teacher said to five students, Alan, Bob, Carl, Dick and Eason, 'I have written down a five-digit number N which is made up of five different digits. I will let Alan see the ten thousands and thousands digits of N, let Bob see the thousands and hundreds digits, let Carl see the hundreds and tens digits, let Dick see the tens and unit digits and let Eason see the unit and ten thousands digits.' The teacher then let each student know two digits of N as said, and then everybody sat in a circle and started the following conversation.

'Raise your hands if you know a prime factor of N,' said the teacher, and then two students raised their hands.

'Raise your hands if you know a prime factor of N,' asked the teacher again, and this time three students raised their hands.

'Raise your hands if you know a composite factor of N,' the teacher continued, and then two students raised their hands.

'Raise your hands if you know two composite factors of N,' said the teacher, but no student raised their hands.

Then the teacher asked, 'who knows the value of N?'

One student said, 'I know. N is a multiple of 59.'

Assuming all students to be clever (which means that deductions can be made whenever sufficient information is given), find N.

全卷完

END OF PAPER