第九屆培正數學邀請賽

9th Pui Ching Invitational Mathematics Competition

決賽（中二組）

Final Event (Secondary 2)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

(a) 本卷共設 20 題，總分爲 100 分。

There are 20 questions in this paper and the total score is 100.

(b) 除特別指明外，本卷內的所有數均爲十進制。

Unless otherwise stated, all numbers in this paper are in decimal system.

(c) 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

(d) 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

(e) 不得使用計算機。

The use of calculators is not allowed.

(f) 本卷的附圖不一定依比例繪成。

The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

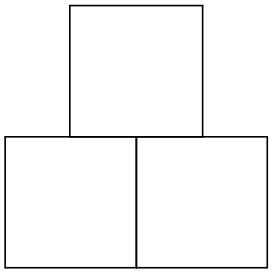
1. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1] = 1$、$[6.9] = 6$ 和 $[5] = 5$。求 $\left[\frac{1004\times1997}{2010}\right]+\left[\frac{1006\times1997}{2010}\right]$ 的值。

   Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1] = 1$, $[6.9] = 6$ and $[5] = 5$. Find the value of $\left[\frac{1004\times1997}{2010}\right]+\left[\frac{1006\times1997}{2010}\right]$.

2. 一家百貨店進行「買二送一」推廣，顧客每次購買兩件貨品，可獲贈一件貨品，但贈品的價值須分別不高於購買的兩件貨品的價值。美詩希望得到 15 件價值分別是 1 元、2 元、…、15 元的貨品，則她最少要付多少元？

   A department store has launched a 'buy 2 get 1 free' promotion. Each time when one buys 2 articles, one is entitled to get an article for free, subject to the condition that the value of the free article must not exceed each of the two purchased articles. If Macy wants to get 15 articles valued at 1 dollar, 2 dollars, …, 15 dollars respectively, what is the minimum amount (in dollars) that she has to pay?

3. 圖中所示的是一個周界爲 8 的軸對稱圖形，它由三個邊長爲 1 的正方形拼合而成。若 X 和 Y 是周界上的兩點，且 X 和 Y 的直線距離是 d，求 d 的最大可能值。

   The figure shown has perimeter 8, possesses reflectional symmetry and is made up of 3 squares of side length 1. If X and Y are two points on the perimeter and the straight-line distance between X and Y is d, find the greatest possible value of d.

4. 設 n 爲大於 1 的整數。若 n 個連續整數之和爲 2010，求 n 的最小可能值。

   Let n be an integer greater than 1. If the sum of n consecutive integers is 2010, find the smallest possible value of n.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 某五位數 n 的五個數字分別是 1、3、4、5 和 6。已知 n 是 11 的倍數，求 n 的最小可能值。

   The five digits of a five-digit number n are 1, 3, 4, 5 and 6. Given that n is a multiple of 11, find the smallest possible value of n.

6. 詩雅有兩張全等的長方形紙。她用兩種不同的方法將它們拼成一個大長方形。這兩個大長方形的對角線的長度分別是 19 和 22。求長方形紙的對角線的長度。

   Alice had two congruent sheets of paper in rectangular shape. She built a big rectangle with them by two different methods. It is known that the lengths of the diagonals of two big rectangles are 19 and 22. Find the length of the diagonal of the rectangular sheet of paper.

7. 在所示的乘式中，每個字母代表一個由 0 至 9 的不同數字。求乘數 PQPP 所代表的四位數。

   In the multiplication shown, each letter represents a different digit from 0 to 9. Find the four-digit number represented by the multiplier PQPP.

$$\begin{array}{rr} & \text{P Q P P P} \\ \times & \text{P Q P P} \\ \hline \text{R P P P P P P R P} & \end{array}$$

8. 小燊有 10 張分別寫上 0 至 9 的咭片。現在他要利用這 10 張咭片組成 5 個能被 9 整除的兩位數並把它們從小至大排列。他有多少種不同的方法組成這些兩位數？

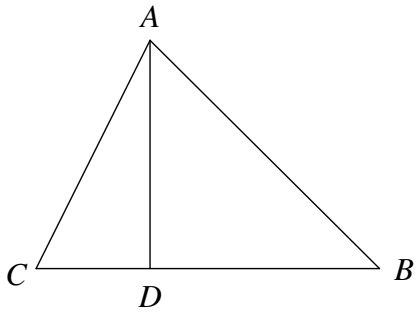
   Eric has 10 cardboards with the numbers 0 to 9 written. He has to form 5 two-digit numbers which are divisible by 9 with these cardboards and arrange them in ascending order. In how many different ways can he form these two-digit numbers?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 圖中，ABC 是銳角三角形，D 是 A 到 BC 的垂足。若 $AD=8$、$BC=11$ 且 $AB^2-AC^2=66$，求 ΔACD 的面積。

In the figure, ABC is an acute-angled triangle. D is the foot of perpendicular from A to BC. If $AD=8$, $BC=11$ and $AB^2-AC^2=66$, find the area of ΔACD.

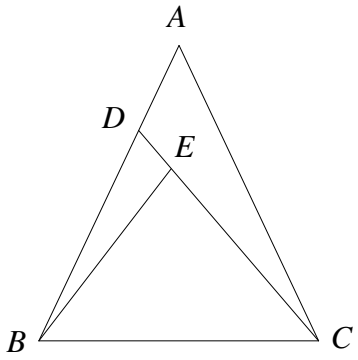


10. 若要從首 2010 個正整數中選兩個不同的數，使得它們的最大公因數可被 30 整除，問有多少個不同的選擇？

If two different numbers are to be chosen from the first 2010 positive integers such that their H.C.F. is divisible by 30, how many different choices are there?

11. 圖中，E 是 ΔABC 內的一點，CE 延長後交 AB 於 D。若 $AB=6$、$BC=5$、$CA=7$ 且 $\Delta ABC\sim\Delta CEB$，求 $CD\times CE$。

In the figure, E is a point inside ΔABC and CE is produced to meet AB at D. If $AB=6$, $BC=5$, $CA=7$ and $\Delta ABC\sim\Delta CEB$, find $CD\times CE$.



12. 設 $[x]$ 代表不超過 x 的最大整數，例如 $[1.1]=1$、$[6.9]=6$ 和 $[5]=5$。若 $\lfloor 0.1n\pi\rfloor=\lfloor 0.1(n-1)\pi\rfloor$，其中 n 是不超過 100 的正整數，問 n 有多少個不同的可能值？

Let $[x]$ denote the greatest integer not exceeding x. For example, $[1.1]=1$, $[6.9]=6$ and $[5]=5$. If $\lfloor 0.1n\pi\rfloor=\lfloor 0.1(n-1)\pi\rfloor$ where n is a positive integer not exceeding 100, how many different possible values of n are there?

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 已知 n 是正整數，它被 7 除時餘 3，被 11 除時餘 5，被 13 除時餘 6，被 101 除時餘 50。求 n 的最小可能值。

Let n be a positive integer which leaves a remainder of 3 when divided by 7, a remainder of 5 when divided by 11, a remainder of 6 when divided by 13 and a remainder of 50 when divided by 101. Find the smallest possible value of n.

14. 求 $\frac{1}{10}+\frac{1}{100}+\frac{2}{1000}+\frac{3}{10000}+\frac{5}{100000}+\cdots$ 的值。（各項的分子是數列 1, 1, 2, 3, 5, 8, 13, … 的連續項，此數列的首兩項爲 1，之後每項都是前兩項之和。）

Find the value of $\frac{1}{10}+\frac{1}{100}+\frac{2}{1000}+\frac{3}{10000}+\frac{5}{100000}+\cdots$. (The numerators of the summands are consecutive terms of the sequence 1, 1, 2, 3, 5, 8, 13, …, in which the first two terms are 1 and each subsequent term is the sum of the two preceding terms.)

15. 嘉美想了四個正整數 a、b、c 和 d。她求得 $a+b$、$a+c$、$a+d$、$b+c$、$b+d$ 和 $c+d$ 的值後，發現其中四個是 41、70、82 和 98。求 $a+b+c+d$ 的值。

Gloria thought of four positive integers a, b, c and d. After computing the values of $a+b$, $a+c$, $a+d$, $b+c$, $b+d$ and $c+d$, she found that four of these values were 41, 70, 82 and 98. Find the value of $a+b+c+d$.

16. 某班同學進行考試，試卷共設 5 題，題號分別是 1 至 5。對任意兩題，同時答對這兩題的人數都剛好是該兩題的題號之和（例如有 $1+3=4$ 名同學同時答對第 1 題和第 3 題，有 $2+5=7$ 名同學同時答對第 2 題和第 5 題，如此類推）。問該班的學生最少有幾人？

A class of students sat for an examination. There were 5 questions, numbered 1 to 5. For any two questions, the number of students who answered both of them correctly is equal to the sum of the two question numbers. (For example, $1+3=4$ students answered both Questions 1 and 3 correctly, $2+5=7$ students answered both Questions 2 and 5 correctly, and so on.) What is the least number of students in the class?

第 17 至第 20 題，每題 7 分。

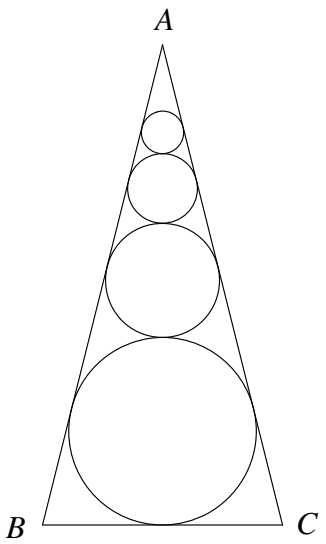
Questions 17 to 20 each carries 7 marks.

17. 一個袋子中共有 100 枚硬幣，其中每個都是二元、五元或十元硬幣。若這 100 個硬幣共值 n 元，問 n 有多少個不同的可能值？

    There are 100 coins in a bag, each of denomination 2 dollars, 5 dollars or 10 dollars. If the total value of the 100 coins is n dollars, how many different possible values of n are there?

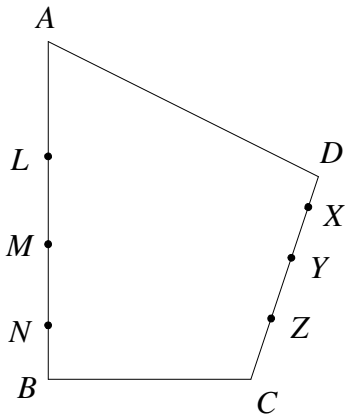
18. 圖中，ABC 是等腰三角形，其中 $AB = AC$。四個圓之間任何連續的兩個均互相外切，且每個均與 AB 和 AC 相切，最大的一個圓同時與 BC 相切。當中最小和最大圓的半徑分別是 8 和 27。已知 ΔABC 的面積可寫成 $p\sqrt{q}$，其中 p、q 爲正整數，求 p 的最大可能值。

    In the figure, ΔABC is isosceles with $AB = AC$. The four circles are such that any two consecutive circles are externally tangent to each other, each is tangent to AB and AC, and the largest one is tangent to BC as well. The radii of the smallest and largest circles are 8 and 27 respectively. Given that the area of ΔABC can be expressed as $p\sqrt{q}$ where p, q are positive integers, find the greatest possible value of p.



19. 圖中，四邊形 $ABCD$ 的面積爲 180。L、M、N 是 AB 上的點，使得 $AL:LM:MN:NB=4:3:2:1$；X、Y、Z 則是 DC 上的點，使得 $DX:XY:YZ:ZC=2:4:7:7$。若 ΔAXD 和四邊形 $AMYD$ 的面積分別是 12 和 78，求四邊形 $LNZX$ 的面積。

    In the figure, $ABCD$ is a quadrilateral with area 180. L, M, N are points on AB such that $AL:LM:MN:NB=4:3:2:1$, while X, Y, Z are points on DC such that $DX:XY:YZ:ZC=2:4:7:7$. If the areas of ΔAXD and quadrilateral $AMYD$ are 12 and 78 respectively, find the area of quadrilateral $LNZX$.



20. 在課室裡，老師對五名學生小陳、小李、小張、小王和小何說：「我寫下了一個五位數 N，它由五個不同的數字組成。我會讓小陳看 N 的萬位和千位，讓小李看千位和百位，讓小張看百位和十位，讓小王看十位和個位，並讓小何看個位和萬位。」之後老師如所述般讓每名學生知道 N 的兩個數字，然後各人圍圈而坐，展開了以下的對話。

老師說：「知道 N 的一個質因數的請舉手」，之後有兩名學生舉手。

老師再問：「知道 N 的一個質因數的請舉手」，這次有三名學生舉手。

老師說：「知道 N 的一個合成數因數的請舉手」，之後有兩名學生舉手。

老師又說：「知道 N 的兩個合成數因數的請舉手」，但沒有學生舉手。

之後老師問：「誰知道 N 的值？」

其中一名學生說：「我知道，N 是 59 的倍數呢。」

假設所有學生都是聰明的（即有足夠資料便能作出推論），求 N。

In a classroom, the teacher said to five students, Alan, Bob, Carl, Dick and Eason, 'I have written down a five-digit number N which is made up of five different digits. I will let Alan see the ten thousands and thousands digits of N, let Bob see the thousands and hundreds digits, let Carl see the hundreds and tens digits, let Dick see the tens and unit digits and let Eason see the unit and ten thousands digits.' The teacher then let each student know two digits of N as said, and then everybody sat in a circle and started the following conversation.

'Raise your hands if you know a prime factor of N,' said the teacher, and then two students raised their hands.

'Raise your hands if you know a prime factor of N,' asked the teacher again, and this time three students raised their hands.

'Raise your hands if you know a composite factor of N,' the teacher continued, and then two students raised their hands.

'Raise your hands if you know two composite factors of N,' said the teacher, but no student raised their hands.

Then the teacher asked, 'who knows the value of N?'

One student said, 'I know. N is a multiple of 59.'

Assuming all students to be clever (which means that deductions can be made whenever sufficient information is given), find N.

全卷完

END OF PAPER