第八屆培正數學邀請賽

8th Pui Ching Invitational Mathematics Competition

決賽（高中組）

Final Event (Senior Secondary)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

1. 本卷共設 20 題，總分爲 100 分。

   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。

   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。

   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。

   The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 下表詳列了某城市的用水收費。該市的一所理髮店毗鄰一所完全不用水的書店，書店東主決定以每立方米 8 元的價格把水賣給理髮店。假設除水費外，書店並沒有其他成本，那麼書店賣水每月最多可獲多少元的利潤？

   The following table lists the water charges of a city. A barber shop in the city is next to a bookstore which does not consume any water. The owner of the bookstore decides to sell water to the barber shop at a price 8 dollars per cubic metre. Assume that there is no cost for the bookstore except the water charges, what is the maximum possible monthly profit (in dollars) for the bookstore by selling water?

每月用水收費表	Table for monthly fee of water supply
首 10 立方米 .... 免費	The first 10 cubic metres.................................... Free
次 20 立方米 .... 每立方米 2 元	The next 20 cubic metres ...2 dollars per cubic metre
次 30 立方米 .... 每立方米 5 元	The next 30 cubic metres ...5 dollars per cubic metre
次 40 立方米 .... 每立方米 9 元	The next 40 cubic metres ...9 dollars per cubic metre
.... 以後每立方米 15 元	.... 15 dollars per cubic metre thereafter

2. 設 x、y、z 爲某等差數列的連續項。若 $3^x = 5$ 而 $3^y = 10$。求 3^z。

   Let x, y, z be three consecutive terms of an arithmetic sequence. If $3^x = 5$ and $3^y = 10$, find 3^z.

3. 已知 $f(0) = 6$，且對任意實數 x 皆有 $f(x) \neq 2$ 及 $f\left(x + \frac{1}{3}\right) = \frac{2f(x) - 5}{2f(x) - 4}$。求 $f(2009)$ 的值。

   Suppose $f(0) = 6$, and that $f(x) \neq 2$ and $f\left(x + \frac{1}{3}\right) = \frac{2f(x) - 5}{2f(x) - 4}$ for all real numbers x. Find the value of $f(2009)$.

4. 某村莊有 2009 人，其中 x 人永遠說真話，其餘的人永遠說謊話。某天，那 2009 人圍成一圈，然後齊聲說：「和我相鄰的兩個人當中，最少一個是說謊話的人」。求 x 的最大可能值。

   There are 2009 people in a village. Among them x people always tell the truth and the others always lie. One day, the 2009 people formed a circle and said together, ‘of the two people next to me, at least one is a liar’. Find the greatest possible value of x.

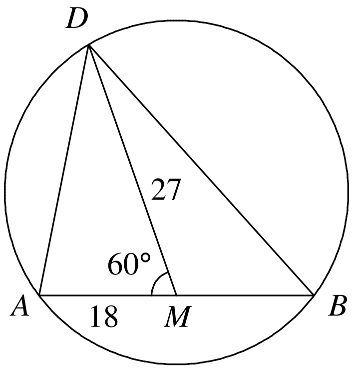
第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 設 $a_1 = 0$，且對任意大於 1 的整數 n 皆有 $a_n = a_{n-1} + \log_{2009}\left(1-\frac{1}{n}\right)$。求 a_{2009} 的值。

Let $a_1 = 0$ and $a_n = a_{n-1} + \log_{2009}\left(1-\frac{1}{n}\right)$ for any integer n greater than 1. Find the value of a_{2009}.

6. 圖中，AB 是圓的弦，M 是 AB 的中點，D 是圓周上的一點。若 $MA = 18$、$MD = 27$、$\angle AMD = 60°$，且圓的面積爲 $k\pi$，求 k。

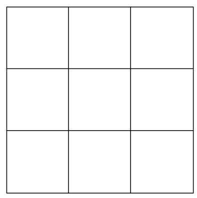


In the figure, AB is a chord of the circle, M is the mid-point of AB and D is a point on the circumference. If $MA = 18$, $MD = 27$ and $\angle AMD = 60°$, and that the area of the circle is $k\pi$, find k.

7. 已知 $\begin{cases} 50\sin^2 t + 5m\sin t + (4m-41) = 0 \\ 50\cos^2 t + 5m\cos t + (4m-41) = 0 \end{cases}$，而且 $\tan t \neq 1$，求 m 的值。

Given that $\begin{cases} 50\sin^2 t + 5m\sin t + (4m-41) = 0 \\ 50\cos^2 t + 5m\cos t + (4m-41) = 0 \end{cases}$ and $\tan t \neq 1$, find the value of m.

8. 圖中有一個 3×3 的方格表。現要把每格塗上紅色或藍色，使得當方格表繞中心旋轉 90°、180° 或 270° 時，方格表的顏色規律都和原來相同。問有多少種不同的塗色方法？

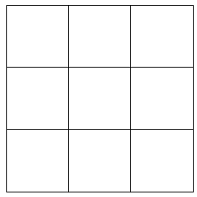


The figure shows a 3×3 grid. Each cell is to be painted red or blue, in a way such that the colour pattern of the grid would remain the same when the grid is rotated by its centre through 90°, 180° or 270°. How many different colourings are possible?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

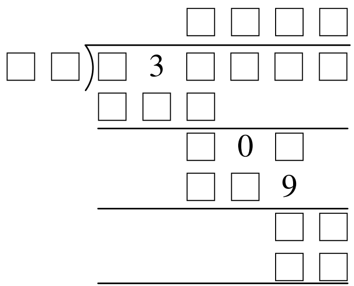
9. 惠英在圖中的 3×3 方格表裏每小格填進一個正整數（容許重覆）。填好後，她發現三橫行的和都是互不相同的合成數，而三直行的和都是互不相同的質數。若方格表裏的九個數之和是 n，求 n 的最小可能值。



Amy filled every cell of a 3×3 grid in the figure with a positive integer (repetition is allowed). After that, she found that the three sums of rows are pairwise distinct composite numbers while the three sums of columns are pairwise distinct prime numbers. If the sum of the nine numbers in the grid is n, find the smallest possible value of n.

10. 圖中顯示一條除式，但當中有些數字留空了。求該六位的被除數。

The figure shows a division, but some digits are left out. Find the six-digit dividend.



11. 已知 $\left(n+\dfrac{x}{3n^2}\right)^3 = n^3 + x + \dfrac{x^2}{3n^3} + \dfrac{x^3}{27n^6}$。求 $\sqrt[3]{8030}$ 的近似值，準確至小數點後三位。

Given $\left(n+\dfrac{x}{3n^2}\right)^3 = n^3 + x + \dfrac{x^2}{3n^3} + \dfrac{x^3}{27n^6}$. Find the approximate value of $\sqrt[3]{8030}$ correct to 3 decimal places.

12. 已知 $\lim\limits_{n\to\infty}\dfrac{1^7+2^7+3^7+\cdots+n^7}{n^8}=\dfrac{1}{8}$，求 $\lim\limits_{n\to\infty}\dfrac{1^7+3^7+5^7+\cdots+(4n+2009)^7}{n^8}$ 的值。

Given $\lim\limits_{n\to\infty}\dfrac{1^7+2^7+3^7+\cdots+n^7}{n^8}=\dfrac{1}{8}$, find the value of

$$\lim_{n\to\infty}\frac{1^7+3^7+5^7+\cdots+(4n+2009)^7}{n^8}.$$

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 現容許在整數 11111111 的兩個連續「1」字之間加上「+」號來組成不同的正整數，例如：我們可以得到 $1111+1111=2222$ 和 $1+11+1+1+11+1=26$ 。可以這樣得出來的正整數稱爲「好數」（11111111 本身亦是「好數」）。問共有多少個不同的「好數」？

    From the number 11111111, one is allowed to insert the symbol '+' between two consecutive 1's to form various positive integers. For instance, one may get $1111+1111=2222$ and $1+11+1+1+11+1=26$. Positive integers which can be formed in this way are said to be 'good'. (The number 11111111 itself is also 'good'.) How many different 'good' positive integers are there?

14. 小琪到 X 國旅行 7 天。該國有四個城市，A、B、C、D。每天，小琪都會遊覽其中一個城市，而且不會連續兩天遊覽同一城市。已知小琪在第 1 天和第 7 天都是遊覽城市 A，問小琪這 7 天的行程有多少個不同的可能性？

    Vicky goes for a 7-day trip in Country X. There are four cities, A, B, C and D, in the country. Each day Vicky visits one city, and she will not visit the same city on two consecutive days. Given that Vicky visits City A on both Day 1 and Day 7, how many different possible itineraries are there for her 7-day trip?

15. 如果某數由左至右和由右至左看皆相同，我們稱這個數是「回文數」。例如 3883、12321 和 25052 都是「回文數」。求 2600 和 3000 之間的一個整數 n，使得 n^2 是「回文數」。

    If a number reads the same from left to right as from right to left, it is called a 'palindrome'. For example, 3883, 12321 and 25052 are 'palindromes'. Find an integer n between 2600 and 3000 for which n^2 is a 'palindrome'.

16. 若函數 $f(x)$ 滿足 $f(0)=5$ 和 $f'(x)=6x+\sqrt{2+x^2}\sin^2 x$ ，求 $\int_{-2}^{2} f(x)\mathrm{d}x$ 。

    If the function $f(x)$ satisfies $f(0)=5$ and $f'(x)=6x+\sqrt{2+x^2}\sin^2 x$, find $\int_{-2}^{2} f(x)\mathrm{d}x$.

第 17 至第 20 題，每題 7 分。

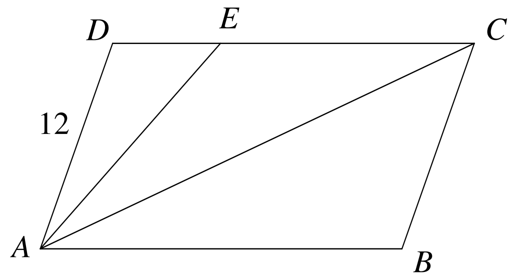
Questions 17 to 20 each carries 7 marks.

17. 若要把數字 1、3、3、8、8、8 重新排列成一個可被 7 整除的六位數，共有多少個不同的排列？

    If the digits 1, 3, 3, 8, 8, 8 are to be permuted to form a six-digit number divisible by 7, how many different permutations are possible?

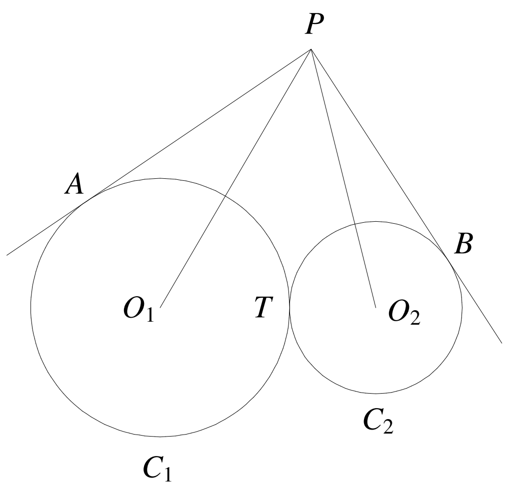
18. 圖中，*ABCD* 是平行四邊形，其中 $\angle CAD = 2\angle CAB$。$\angle CAD$ 的角平分線交 *CD* 於 *E*。若 $AD = 12$ 而 $CE = DE + 2$，求 *AC*。

    In the figure, *ABCD* is a parallelogram with $\angle CAD = 2\angle CAB$. The bisector of $\angle CAD$ meets *CD* at *E*. If $AD = 12$ and $CE = DE + 2$, find *AC*.



19. 圖中，圓 C_1 和 C_2 外切於 *T*，兩圓分別以 O_1 和 O_2 爲圓心，半徑分別爲 3 和 2。*A* 和 *B* 分別是 C_1 和 C_2 上的點，使得 *PA* 切 C_1 於 *A*，*PB* 切 O_2 於 *B*。若 $\angle APB = 2\angle O_1PO_2$ 而 ΔO_1PO_2 的面積是 15，求 ΔAPB 的面積。

    In the figure, the circles C_1 and C_2 are externally tangent at *T*. They have centres O_1 and O_2 respectively and radii 3 and 2 respectively. *A* and *B* are points on C_1 and C_2 respectively such that *PA* is tangent to C_1 at *A* and *PB* is tangent to O_2 at *B*. If $\angle APB = 2\angle O_1PO_2$ and ΔO_1PO_2 has area 15, find the area of ΔAPB.



20. 若 $a_1 = 1$，且對所有 $n \geq 1$ 均有 $a_{n+1} > a_n$ 和 $\dfrac{{a_n}^2 + {a_{n+1}}^2 + 2007}{a_n a_{n+1}} = 2009$，求 a_5 除以 2006 時的餘數。

If $a_1 = 1$, and for any $n \geq 1$ we have $a_{n+1} > a_n$ and $\dfrac{{a_n}^2 + {a_{n+1}}^2 + 2007}{a_n a_{n+1}} = 2009$, find the remainder when a_5 is divided by 2006.

全卷完

END OF PAPER