第八屆培正數學邀請賽

8th Pui Ching Invitational Mathematics Competition

決賽（中四組）

Final Event (Secondary 4)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

1. 本卷共設 20 題，總分爲 100 分。

   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。

   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

   Put your answers on the space provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。

   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。

   The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 某凸 n 邊形有 $2009n$ 條對角線。求 n。

   A convex n-sided polygon has $2009n$ diagonals. Find n.

2. 設 m 是正整數，m 的數字之和是 n。若 n 的數字之和是 4 而且 m、n 的所有數字都不是 0，問 m 最多有多少個數字？

   Let m be a positive integer with sum of digits n. If the sum of digits of n is 4 and all digits of m and n are non-zero, what is the maximum number of digits that m may have?

3. 若實數 x 滿足 $x=\cos x$，求 $\frac{1-2x^2}{\cos(2\cos x)}$ 的值。

   If a real number x satisfies $x=\cos x$, find the value of $\frac{1-2x^2}{\cos(2\cos x)}$.

4. 某村莊有 2009 人，其中 x 人永遠說真話，其餘的人永遠說謊話。某天，那 2009 人圍成一圈，然後齊聲說：「和我相鄰的兩個人當中，最少一個是說謊話的人」。求 x 的最大可能値。

   There are 2009 people in a village. Among them x people always tell the truth and the others always lie. One day, the 2009 people formed a circle and said together, ‘of the two people next to me, at least one is a liar’. Find the greatest possible value of x.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 若 n 和 $\log 1250n$ 均為正整數，求 n 的最小可能値。

   If n and $\log 1250n$ are both positive integers, find the smallest possible value of n.

6. 下表詳列了某城市的用水收費。該市的一家水果店某月的用水量比一家雜貨店多 46 立方米，結果水果店需比雜貨店多付 400 元水費。如果雜貨店該月的用水量爲 m 立方米，求 m 的值。

The following table lists the water charges of a city. The water usage of a fruit shop in the city is 46 cubic meters more than that of a grocery in a certain month, and the fruit shop turns out to pay 400 dollars more than the grocery on water charges. If the water usage of the grocery that month is m cubic metres, find the value of m.

每月用水收費表	Table for monthly fee of water supply
首 10 立方米 ....免費	The first 10 cubic metres.................................... Free
次 20 立方米 .... 每立方米 2 元	The next 20 cubic metres ...2 dollars per cubic metre
次 30 立方米 .... 每立方米 5 元	The next 30 cubic metres ...5 dollars per cubic metre
次 40 立方米 .... 每立方米 9 元	The next 40 cubic metres ...9 dollars per cubic metre
.... 以後每立方米 15 元	.... 15 dollars per cubic metre thereafter

7. 現有一個正八邊形，若從中任意選三個頂點來組成三角形，則該三角形是直角三角形的概率是多少？

A triangle is formed by picking three vertices form a regular octagon at random. What is the probability that the triangle is right-angled?

8. 已知 $\begin{cases} 50\sin^2 t + 5m\sin t + (4m-41) = 0 \\ 50\cos^2 t + 5m\cos t + (4m-41) = 0 \end{cases}$，而且 $\tan t \neq 1$，求 m 的值。

Given that $\begin{cases} 50\sin^2 t + 5m\sin t + (4m-41) = 0 \\ 50\cos^2 t + 5m\cos t + (4m-41) = 0 \end{cases}$ and $\tan t \neq 1$, find the value of m.

第 9 至第 12 題，每題 5 分。

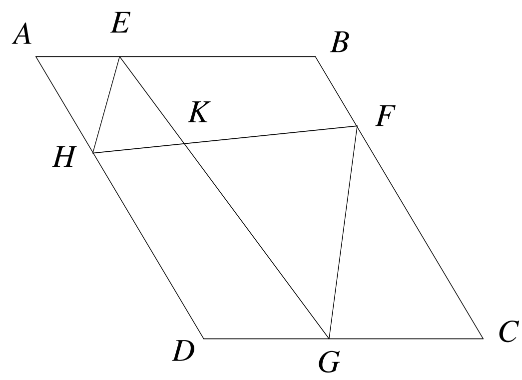
Questions 9 to 12 each carries 5 marks.

9. 已知 $\left(n + \frac{x}{2n}\right)^2 = n^2 + x + \frac{x^2}{4n^2}$。求最接近 $\left(\sqrt{4956} - \sqrt{3612}\right)^2$ 的整數。

Given $\left(n + \frac{x}{2n}\right)^2 = n^2 + x + \frac{x^2}{4n^2}$. Find the integer closest to $\left(\sqrt{4956} - \sqrt{3612}\right)^2$.

10. 圖中，*ABCD* 是邊長為 17 的菱形。*E*、*F*、*G*、*H* 分別是 *AB*、*BC*、*CD* 和 *DA* 上的點，使得 $AE = 4$、$BF = 5$、$CG = 8$、$DH = 11$。設 *EG* 與 *FH* 交於 *K*。若 ΔKFG 的面積是 ΔKEH 的面積的 k 倍，求 k。

In the figure, *ABCD* is a rhombus with side length 17. *E*, *F*, *G*, *H* are points on *AB*, *BC*, *CD* and *DA* respectively such that $AE = 4$, $BF = 5$, $CG = 8$ and $DH = 11$. Let *EG* and *FH* intersect at *K*. If the area of ΔKFG is k times that of ΔKEH, find k.

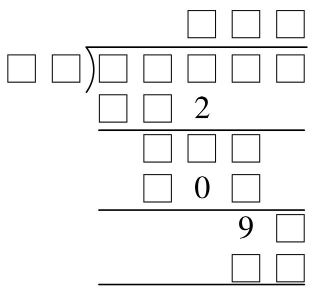


11. 某星球是半徑為 $\frac{10}{\pi}$ 公里的球體，它使用與地球相同的方向表示法。俊熙在該星球的赤道上的一點出發，沿赤道繞星球轉四分之一個圈後往北走 1 公里，最後在星球表面上沿最短的路徑返回起點。他全程共走了多少公里？

A planet is a sphere of radius $\frac{10}{\pi}$ km. It adopts the same representation of directions as on the Earth. Bob started his travel from a point on the equator of the planet. He orbited a quarter of the planet along the equator, and then went north for 1 km. Finally, he returned to his starting point via the shortest path on the surface of the planet. What is the total distance he travelled (in km)?

12. 圖中顯示一條除式，但當中有些數字留空了。求該五位的被除數。

The figure shows a division, but some digits are left out. Find the five-digit dividend.



第 13 至第 16 題，每題 6 分。

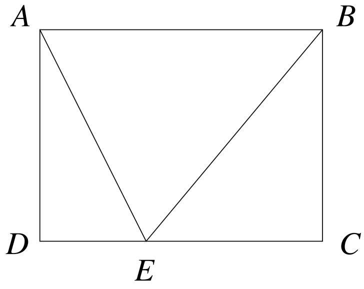
Questions 13 to 16 each carries 6 marks.

13. 若要選兩個正整數（可以相同）使得它們的最小公倍數是 900，有多少個不同的選擇？

If two (possibly equal) positive integers are to be chosen such that their L.C.M. is 900, how many different choices are there?

14. 圖中，$ABCD$ 是長方形，E 是 CD 上的一點，而 BC 的長度爲 x。已知當 E 在 CD 上（包括端點）移動時，$AE+BE$ 的值最小是 14，最大是 16。求 x 所有可能值之和。

In the figure, $ABCD$ is a rectangle, E is a point on CD and the length of BC is x. It is known that as E moves along CD (including the end-points), the minimum value of $AE+BE$ is 14 and the maximum value is 16. Find the sum of all possible values of x.



15. 如果對任意正整數 n 皆有 $a_{n+k}=a_n$，則我們說 k 是無窮數列 $a_1, a_2, \ldots$ 的一個週期。例如：數列 1, 3, 4, 1, 3, 4, 1, 3, 4, … 的一個週期是 3。有多少個數列的每項皆是 1、2、3、4 或 5，且最小週期是 6？

If an infinite sequence $a_1, a_2, \ldots$ satisfies $a_{n+k}=a_n$ for all positive integers n, we say that k is a period of the sequence. For instance, 3 is a period of the sequence 1, 3, 4, 1, 3, 4, 1, 3, 4, … How many sequences have each term being 1, 2, 3, 4 or 5 and have smallest period 6?

16. 已知 a、b、c、d、e、f、g 是七個互不相同的正整數，其中 $a<b$、$a<c$ 而 $d<e<f$。若把七個數從小至大排列，有多少個不同的可能結果？

Let a, b, c, d, e, f, g be seven pairwise different positive integers such that $a<b$, $a<c$ and $d<e<f$. If the seven numbers are arranged in ascending order, how many different results are possible?

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 若 $a_1=1$，且對所有 $n\geq 1$ 均有 $a_{n+1}>a_n$ 和 $\dfrac{a_n^{\,2}+a_{n+1}^{\,2}+2007}{a_n a_{n+1}}=2009$，求 a_5 除以 2006 時的餘數。

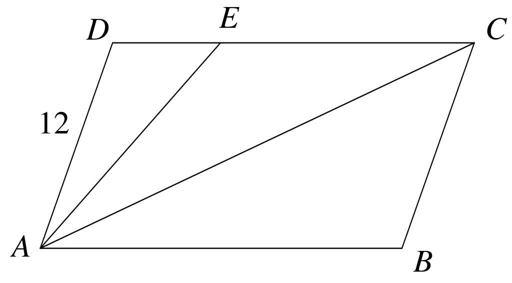
If $a_1=1$, and for any $n\geq 1$ we have $a_{n+1}>a_n$ and $\dfrac{a_n^{\,2}+a_{n+1}^{\,2}+2007}{a_n a_{n+1}}=2009$, find the remainder when a_5 is divided by 2006.

18. 求 $1^3+11^3+21^3+\cdots+20091^3+20101^3$ 除以 2009 時的餘數。

Find the remainder when $1^3+11^3+21^3+\cdots+20091^3+20101^3$ is divided by 2009.

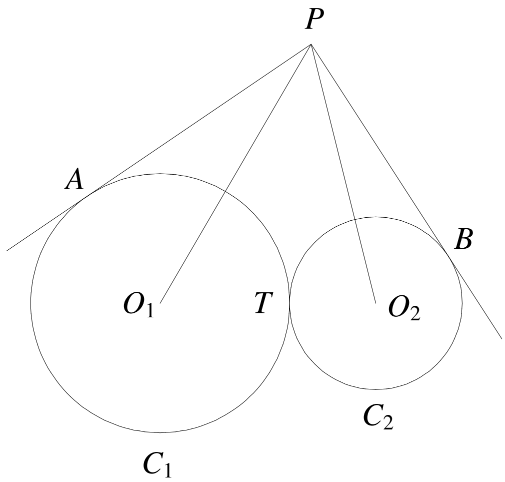
19. 圖中，*ABCD* 是平行四邊形，其中 $\angle CAD = 2\angle CAB$。$\angle CAD$ 的角平分線交 *CD* 於 *E*。若 $AD = 12$ 而 $CE = DE + 2$，求 *AC*。

In the figure, *ABCD* is a parallelogram with $\angle CAD = 2\angle CAB$. The bisector of $\angle CAD$ meets *CD* at *E*. If $AD = 12$ and $CE = DE + 2$, find *AC*.



20. 圖中，圓 C_1 和 C_2 外切於 *T*，兩圓分別以 O_1 和 O_2 爲圓心，半徑分別爲 3 和 2。*A* 和 *B* 分別是 C_1 和 C_2 上的點，使得 *PA* 切 C_1 於 *A*，*PB* 切 O_2 於 *B*。若 $\angle APB = 2\angle O_1PO_2$ 而 ΔO_1PO_2 的面積是 15，求 ΔAPB 的面積。

In the figure, the circles C_1 and C_2 are externally tangent at *T*. They have centres O_1 and O_2 respectively and radii 3 and 2 respectively. *A* and *B* are points on C_1 and C_2 respectively such that *PA* is tangent to C_1 at *A* and *PB* is tangent to O_2 at *B*. If $\angle APB = 2\angle O_1PO_2$ and ΔO_1PO_2 has area 15, find the area of ΔAPB.



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